Futures

MATH 472 *Financial Mathematics*

J Robert Buchanan

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Objectives

In this lesson we will learn:

- the definitions of financial instruments known as futures contracts,
- the differences between forward contracts and futures contracts,
- how futures are traded,
- how to price futures in an arbitrage-free setting,
- how to manage a portfolio of futures.

Futures

Futures are similar to forward contracts with the following differences:

- Futures are traded in exchanges, while forward contracts can be set up between any two parties.
- Futures are traded in standardized amounts and with standardized maturity dates, whereas forward contracts can be customized to suit the parties involved.
- Forward contracts are settled at expiration while futures are settled daily.
- Futures are usually settled by an exchange of cash between the parties, while a forward contract may involve physical delivery of some commodity (oil, wheat, *etc.*).
- Futures are considered to have less risk of default since the exchange clearinghouse will require deposits from both parties.

Why Futures Exchanges?

- The variety of forward contracts makes it impractical to keep track of all of them, thus standardization is necessary.
- The price of a forward contract changes over time (even if all other terms remain the same), making the tracking of the value of existing contracts difficult.
- The practice of daily settlement of futures contracts makes these contracts more **liquid** (easier to buy and sell).

Example

Specifications for the S&R 500 index futures contract:

Underlying:	S&R 500 index
Where traded:	Chicago Mercantile Exchange
Size:	\$250 $\times$ S&R 500 index
Months:	Mar, Jun, Sep, Dec
Trading ends:	Business day prior to determination of settlement price
Settlement:	Cash-settled, based on opening price of S&R 500 on third Friday of expiration month.

Suppose the S&R 500 index is currently at \$1100. An investor who wishes to take a long position in \$1.1M worth of the index would purchase 4 futures contracts.

$$4 \times 250 \times 1100 = 1,100,000$$

Marking-to-Market

margin: a deposit on a futures contract held by the clearinghouse to insure against default.

maintenance margin: a minimum level of margin (usually a percentage of the value of the futures contract) required by the clearinghouse.

margin call: a request for additional margin funds made by the clearinghouse.

Example: Margin

Suppose Smith takes a long position in 5000 bushels of corn for September delivery at a price of $2.10/bushel.

- The clearinghouse requires a margin of $800 and maintenance margin of $600.
- The next day the price of the futures contract drops to $2.07/bushel.
- The day after that the price of the futures contract drops to $2.05/bushel.

Give an accounting of Smith's margin balance.

Margin Balance

- On the day of purchase of the futures contract, Smith deposits \$800 in a margin account at the clearinghouse.
- The next day the price drops by \$0.03/bushel, thus the clearinghouse subtracts

$$\$0.03 \times 5000 = \$150$$

  from Smith's margin leaving $800 - 150 = \$650$.
- The day after the price drops by \$0.02/bushel, thus the clearinghouse subtracts

$$\$0.02 \times 5000 = \$100$$

  from Smith's margin leaving $650 - 100 = \$550$.
- Smith must either deposit an addition \$50 or close the account and leave with \$550 (on an investment of \$800).

Extended Example (1 of 4)

Suppose a party purchases 500 futures contracts for $10 each. The contracts mature in 7 days. The continuously compounded interest rate is 10%. The clearing house requires a maintenance margin of 20%. What is the initial margin deposit?

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The initial margin deposit will be

$$(500)(10)(0.20) = \$1000.$$

Example (2 of 4)

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- The margin has also increased by the change in the futures price multiplied by the number of contracts.

Example (2 of 4)

Suppose on day 1, the price of a futures contract has increased to \$10.2927. What has changed?

- The margin deposited on day 0 has earned a day's interest.
- The margin has also increased by the change in the futures price multiplied by the number of contracts.

The margin balance is now

$$1000e^{0.10/365} + (10.2927 - 10)500 = \$1146.62.$$

Since $1146.62 > (500)(10.2927)(0.20) = 1029.27$ (maintenance margin), no margin call is issued.

Example (3 of 4)

Daily marking-to-market continues until the futures contract matures.

Day	No. of Contracts	Futures Price	Price Change	Margin Balance	Margin Call
0	500	10.00	—	1,000.00	—
1	500	10.29	0.29	1,146.62	0.00

Example (3 of 4)

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0	500	10.00	—	1,000.00	—
1	500	10.29	0.29	1,146.62	0.00
2	500	11.78	1.49	1,892.57	0.00

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Day	No. of Contracts	Futures Price	Price Change	Margin Balance	Margin Call
0	500	10.00	—	1,000.00	—
1	500	10.29	0.29	1,146.62	0.00
2	500	11.78	1.49	1,892.57	0.00
3	500	10.62	−1.16	1,308.74	0.00
4	500	11.12	0.50	1,560.04	0.00
5	500	8.58	−2.54	293.70	564.66
6	500	6.45	−2.13	−206.35	851.72
7	500	6.27	−0.18	553.18	—

Example (4 of 4)

The profit from the futures contract is calculated at maturity by subtracting the future value of the initial margin from the final margin balance.

$$\text{profit} = 553.18 - (1000)e^{0.10(7/365)} = -447.01$$

Futures Prices

Theorem (Futures-Forwards Equivalence)

If the interest rate is known, the prices of futures and forwards contracts are identical.

Let the price of the forward contract be $F_{0,T}$ and the price of the futures contract be G_t for $t = 0, 1, \ldots, T$. Let the nominal, known interest rate be r compounded n times per year.

Proof (1 of 2)

Consider investment A:

- At $t = 0$ purchase $\left(1 + \frac{r}{n}\right)^{-(T-1)}$ futures contracts,
- At $t = 1$ increase to $\left(1 + \frac{r}{n}\right)^{-(T-2)}$ futures contracts,
- At $t = k$ increase to $\left(1 + \frac{r}{n}\right)^{-(T-k-1)}$ futures contracts,
- At $t = T - 1$ increase to 1 futures contract.

At $t = k + 1$ the profit from the previous period is

$$(G_{k+1} - G_k)\left(1 + \frac{r}{n}\right)^{-(T-k-1)}.$$

Thus the future value of this profit at $t = T$ is $G_{k+1} - G_k$. The total profit is

$$\text{profit}_A = \sum_{k=0}^{T-1}(G_{k+1} - G_k) = G_T - G_0 = S_T - G_0.$$

Proof (2 of 2)

Consider investment B:

- Purchase one forward contract $F_{0,T}$.

The total profit is

$$\text{profit}_B = S_T - F_{0,T}.$$

In an arbitrage-free setting the two profits must be equal.

$$\begin{aligned} \text{profit}_A &= \text{profit}_B \\ S_T - G_0 &= S_T - F_{0,T} \\ G_0 &= F_{0,T} \end{aligned}$$

Homework

- Read Section 6.5
- Exercises: 15–20

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

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