# Gamma Hedging

## MATH 472 *Financial Mathematics*

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# Objectives

In this lesson we will learn to:

- calculate the daily profit/loss on a portfolio of investments,
- form a Delta-neutral portfolio consisting of options and securities,
- form a Delta- and Gamma-neutral portfolio consisting of options and securities

# Self-Financing Portfolios

## Definition

A portfolio consisting of a sold call $C(S, t)$ and a long position in $\Delta$ shares of the underlying security is said to be **self-financing** if once created, no additional cash investments are needed for the portfolio to remain hedged.

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- Consider a sold $K$-strike European call on a security whose current value is $S_0$ and purchased $\Delta_0$ shares of the security.
- Suppose the risk-free interest rate is $r$ and the volatility of the security is $\sigma$.
- **Question**: what moves in security price result in a self-financing portfolio?

# Initial Investment and Future Cost

- At inception the portfolio requires an investment of $C^e(S_0, 0) - (\Delta_0)S_0$ which is borrowed at the risk-free rate $r$.
- At time $t > 0$:
  - the interest charged to the loan is $(C^e(S_0, 0) - (\Delta_0)S_0)(e^{r\,t} - 1)$,
  - the gain on the shares of security held is $\Delta_0(S_t - S_0)$,
  - the gain on the short option is $C^e(S_0, 0) - C^e(S_t, t)$.
- Thus at time $t > 0$ the profit on the portfolio is

$$\Delta_0(S_t - S_0) + (C^e(S_0, 0) - C^e(S_t, t)) + (C^e(S_0, 0) - (\Delta_0)S_0)(e^{r\,t} - 1).$$

- The cost of rebalancing the portfolio at time $t > 0$ is $(\Delta_t - \Delta_0)S_t$.

The portfolio is self-financing if the profit plus the cost of rebalancing is non-negative.

# Market Value of Portfolio

$$\begin{aligned} 0 &\leq (\Delta_t - \Delta_0)S_t + \Delta_0(S_t - S_0) + (C^e(S_0, 0) - C^e(S_t, t)) \\ &\quad + (C^e(S_0, 0) - (\Delta_0)S_0)(e^{r\,t} - 1) \\ C^e(S_t, t) - \Delta_t S_t &\leq (C^e(S_0, 0) - \Delta_0 S_0)e^{r\,t} \end{aligned}$$

The quantity $C^e(S_t, t) - \Delta_t S_t$ is called the **market value** of the portfolio for $t \geq 0$.

# Example

An investor creates a Delta-neutral portfolio after selling $n = 100$ options.

Day 0: A \$40-strike European call option costs \$2.7804, $S_0 = \$40$, $\Delta_0 = 0.5824$, and the risk-free interest rate is $r = 8\%$.

Day 1: A \$40-strike European call option costs \$3.0621, $S_1 = \$40.50$, and $\Delta_1 = 0.6142$.

1. Find the overnight profit for the first day of the portfolio.
2. Find the rebalancing cost for the second day.
3. Is the portfolio self-financing?

# Solution (1 of 2)

- At inception the market value is

$$100(C^e(S_0,0)-\Delta_0 S_0) = 100(2.7804-(0.5824)(40)) = -\$2051.56.$$

- Overnight interest charge:

$$100(C^e(S_0,0)-\Delta_0 S_0)(e^{r\,t}-1) = -2051.56(e^{0.08/365}-1) = -\$0.449706$$

- Gain on securities:

$$100\Delta_0(S_1 - S_0) = 100(0.5824)(40.50 - 40) = \$29.12.$$

- Gain on options:

$$100(C^e(S_0,0) - C^e(S_1,1)) = 100(2.7804 - 3.0621) = -\$28.17.$$

The overnight profit on the portfolio is

$$-0.449706 + 29.12 - 28.17 = \$0.500294.$$

# Solution (2 of 2)

- Rebalancing costs on Day 1:

$$100(\Delta_1 - \Delta_0)S_1 = 100(0.6142 - 0.5824)40.50 = \$128.79.$$

- Since the profit overnight from inception is less than the rebalancing costs, this portfolio is not self-financing.
- The market value of the portfolio in Day 1 is

$$100(C^e(S_1, 1) - \Delta_1 S_1) = 100(3.0621 - (0.6142)(40.50)) = -\$2181.30.$$

# Example

Day 2: A $40-strike European call option costs $2.3282, $S_2 = \$39.25$, and $\Delta_2 = 0.5311$.

1. Find the overnight profit for the second day of the portfolio.
2. Find the rebalancing cost for the third day.

# Solution

- The market value is on Day 1 was $-\$2181.30$.
- Overnight interest charge:

$$2181.30(e^{r\,t} - 1) = -\$0.478146.$$

- Gain on securities:

$$100\Delta_1(S_2 - S_1) = 100(0.6142)(39.25 - 40.50) = -\$76.775.$$

- Gain on options:

$$100(C^e(S_1, 1) - C^e(S_2, 1)) = 100(3.0621 - 2.3282) = \$73.39.$$

- The overnight profit on the portfolio is

$$-0.478146 - 76.775 + 73.39 = -\$3.86315.$$

- Rebalancing cost for the third day:

$$100(\Delta_2 - \Delta_1)S_2 = 100(0.5311 - 0.6142)(39.25) = -\$326.167.$$

# Other Solutions to the Black-Scholes PDE

We have already seen that the values of European Call and Put options satisfy the Black-Scholes PDE.

$$r\,F = F_t + \frac{1}{2}\sigma^2 S^2 F_{SS} + rSF_S$$

Other financial instruments solve the PDE as well (but satisfy different boundary and/or final conditions than the options).

Show that the following are also solutions.

1. $F(S, t) = S$
2. $F(S, t) = A\,e^{r\,t}$

Hence, the security itself and cash are both solutions to the Black-Scholes PDE.

# Delta-Neutral Portfolios

A portfolio consists of a short position in a European call option and a long position in the security (Delta hedged). Thus the net value $\Pi$ of the portfolio is

$$\Pi = C - (\Delta)S = C - \left.\frac{\partial C}{\partial S}\right|_{S=S_0} S.$$

$\Pi$ satisfies the Black-Scholes equation since $C$ and $S$ separately solve it. Thus Delta for the portfolio is

$$\frac{\partial \Pi}{\partial S} = \frac{\partial C}{\partial S} - \left.\frac{\partial C}{\partial S}\right|_{S=S_0}.$$

$\frac{\partial \Pi}{\partial S} \approx 0$ when $S \approx S_0$.

# Taylor Series for $\Pi$

$$
\begin{aligned}
\Pi &= \Pi_0 + \frac{\partial \Pi}{\partial t}(t - t_0) + \frac{\partial \Pi}{\partial S}(S - S_0) + \frac{\partial^2 \Pi}{\partial S^2}\frac{(S - S_0)^2}{2} + \cdots \\
\delta \Pi &= \Theta \delta t + \Delta \delta S + \frac{1}{2}\Gamma(\delta S)^2 + \cdots \\
\delta \Pi &\approx \Theta \delta t + \frac{1}{2}\Gamma(\delta S)^2
\end{aligned}
$$

- $\Theta$ is not stochastic and thus must be retained.
- What about $\Gamma$?

# Gamma Neutral Portfolios

**Recall:** $\Gamma = \dfrac{\partial^2 F}{\partial S^2}$

- Since $\dfrac{\partial^2}{\partial S^2}(S) = 0$ a portfolio cannot be made gamma neutral if it contains only an option and its underlying security.
- Portfolio must include an additional component which depends non-linearly on $S$.
- Portfolio can include two (or more) different types of option dependent on the same security.

# Example (1 of 5)

- Suppose a portfolio contains options with two different strike times written on the same security.
- A firm may sell European call options with a strike time three months and buy European call options on the same security with a strike time of six months.
- Let the number of the 3-month options sold be $n_3$ and the number of the 6-month options purchased be $n_6$.

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- Let the number of the 3-month options sold be $n_3$ and the number of the 6-month options purchased be $n_6$.

  The Gamma of the portfolio would be

  $$\Gamma_\Pi = n_3\Gamma_3 - n_6\Gamma_6,$$

  where $\Gamma_3$ and $\Gamma_6$ denote the Gammas of the 3-month and 6-month options respectively.

# Example (2 of 5)

- Choose $n_3$ and $n_6$ so that $\Gamma_\Pi = 0$.
- Introduce the security so as to make the portfolio Delta neutral.
- **Question:** Why does changing the number of shares of the security in the portfolio affect $\Delta$ but not $\Gamma$?

# Example (2 of 5)

- Choose $n_3$ and $n_6$ so that $\Gamma_\Pi = 0$.
- Introduce the security so as to make the portfolio Delta neutral.
- **Question:** Why does changing the number of shares of the security in the portfolio affect $\Delta$ but not $\Gamma$?

With the proper values of $n_3$ and $n_6$ then

$$\delta\Pi \approx \Theta\,\delta t.$$

# Example (3 of 5)

- Suppose $S = \$100$, $\sigma = 0.22$, and $r = 2.5\%$.
- An investment firm sell 100, 3-month European call options on this security with $K = \$102$.
- The firm buys 6-month European call options on the same security with the same strike price.
- Gamma of the 3-month option is $\Gamma_3 = 0.03618$ and Gamma of the 6-month option is $\Gamma_6 = 0.02563$.
- The portfolio is Gamma neutral in the first quadrant of $n_3 n_6$-space where the equation

$$0.03618 n_3 - 0.02563 n_6 = 0$$

is satisfied.

# Example (4 of 5)

- Since $n_3 = 100$ of the 3-month option were sold,

$$0.03618(100) - (0.02563)n_6 = 0,$$

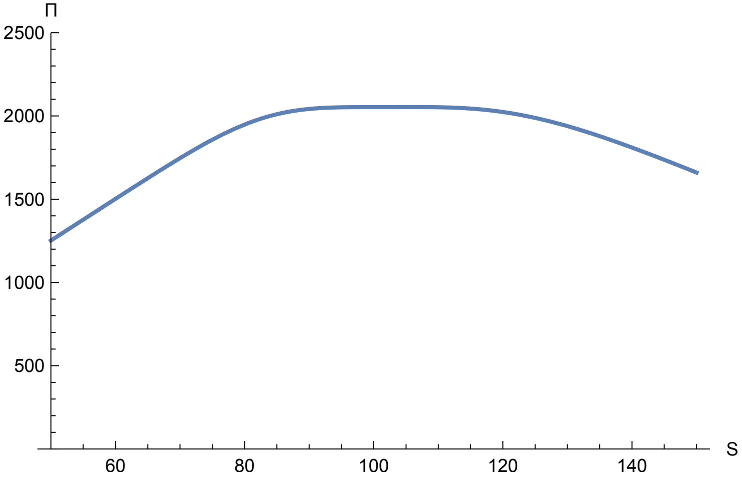
  the portfolio is Gamma neutral if $n_6 = 141.163$ six-month options are purchased.
- Before including the underlying security in the portfolio, the Delta of the portfolio is

$$\begin{aligned} n_3\Delta_3 - n_6\Delta_6 &= (100)(0.4728) - (141.163)(0.5123) \\ &= -25.0368. \end{aligned}$$

- Portfolio can be made Delta neutral if 25.0368 shares of the underlying security are sold short.

# Example (5 of 5)

Over a wide range of values for the underlying security, the value of the portfolio remains nearly constant.



# Example

An investor purchases 100 three-month \$75-strike European put options on a non-dividend-paying security current value is $S_0 = \$76$. The risk-free interest rate is 8% per annum, and the volatility of the price of the security is 50% per annum. The investor also sells $n$ six-month \$75-strike European put options on the same underlying security and then takes a position in $m$ shares of the security. For what values of $m$ and $n$ is the portfolio both Delta- and Gamma-neutral?

# Solution

The value of the portfolio is

$$\Pi = n\,P_6^e - 100P_3^e + m\,S.$$

The Gamma of the portfolio is

$$\begin{aligned}
\Gamma_\Pi &= n\,\Gamma_6 - 100\Gamma_3 \\
0 &= n(0.0140724) - 100(0.0203097) \\
n &= 144.323.
\end{aligned}$$

# Solution

The value of the portfolio is

$$\Pi = n\,P_6^e - 100P_3^e + m\,S.$$

The Gamma of the portfolio is

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\Gamma_\Pi &= n\,\Gamma_6 - 100\Gamma_3 \\
0 &= n(0.0140724) - 100(0.0203097) \\
n &= 144.323.
\end{aligned}$$

The Delta of the portfolio is

$$\begin{aligned}
\Delta_\Pi &= n\,\Delta_6 - 100\Delta_3 + m \\
0 &= (144.323)(-0.371691) - 100(-0.398211) + m \\
m &= 13.8225.
\end{aligned}$$

# Conclusion

- Rho and Vega can be used to hedge portfolios against changes in the interest rate and volatility respectively.
- We have assumed that the necessary options and securities could be bought or sold so as to form the desired hedge.
- If this is not true then a firm or investor may have to substitute a different, but related security or other financial instrument in order to set up the hedge.

# Homework

- Read Sections 10.4
- Exercises: 11–21

# Credits

These slides are adapted from the textbook,

*An Undergraduate Introduction to Financial Mathematics*, 3rd edition, (2012).

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