Elementary Concepts of Interest

MATH 472 *Financial Mathematics*

J Robert Buchanan

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Simple Interest (1 of 2)

Definition

Interest is money paid by a bank or other financial institution to an investor or depositor in exchange for the use of the depositor's money.

Amount of interest is (usually) a fraction (called the **interest rate**) of the initial amount deposited called the **principal amount**.

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Remark: a bank whose interest rate for depositors is the same as its interest rate for borrowers is called an **ideal bank.**

Simple Interest (2 of 2)

Notation:

r: interest rate per unit time

P: principal amount

A: amount due (account balance)

t: time

These quantities are related through the equation:

$$A = P(1 + r\,t).$$

Compound Interest (1 of 2)

Once credited to the investor, the interest may be kept by the investor, and may earn interest itself.

If interest is credited once per year, then after t years the amount due is

$$A = P(1 + r)^t.$$

Compound Interest (2 of 2)

If a portion of the interest is credited after a fraction of a year, then the interest is said to be **compounded.**

If there are n **compounding periods** per year, then in t years the amount due is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Examples (1 of 2)

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Solution:

$$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{n\,t} \\ &= 3104\left(1+\frac{0.0575}{12}\right)^{(3.5)(12)} \\ &\approx 3794.15 \end{aligned}$$

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Effective Interest Rate

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Remark: the effective interest rate is also known as the **effective yield** or simply as the **yield.**

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&\approx 0.0590398
\end{aligned}
$$

Continuous Compounding

What happens as we increase the frequency of compounding?

$$A = \lim_{n \to \infty} P \left(1 + \frac{r}{n}\right)^{n\,t}$$

Evaluate the limit using l'Hôpital's Rule.

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Definition

The amount due for **continuously compounded interest** is

$$A = P\, e^{r\,t}.$$

Example

Suppose \$3585 is deposited in an account which pays interest at an annual rate of 6.15% compounded continuously.

1. Find the amount due after two and one half years.
2. Find the equivalent annual effective simple interest rate.

Solution

1. Amount due:

$$A = Pe^{r\,t} = 3585e^{0.0615(2.5)} \approx 4180.82$$

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2. Effective rate: since $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n - 1 = e^r - 1$ then

$$\begin{aligned} r_e &= e^r - 1 \\ &= e^{0.0615} - 1 \\ &\approx 0.0634305 \end{aligned}$$

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$$P = A\,e^{-rt} \text{ (continuous compounding)}$$

Example

Suppose an investor will receive payments at the end of the next six years in the amounts shown in the table.

Year	1	2	3	4	5	6
Payment	465	233	632	365	334	248

If the interest rate is 3.99% compounded monthly, what is the total present value of the investments?

Solution

t	1	2	3	4	5	6
A_t	465	233	632	365	334	248

$$\begin{aligned} P &= \sum_{t=1}^{6} \left(A_t \left(1 + \frac{0.0399}{12} \right)^{-12t} \right) \\ &= \sum_{t=1}^{6} A_t (0.960949)^t \\ &\approx 2003.01 \end{aligned}$$

Example: Lottery

A lottery has a grand prize of \$10M which is paid in ten payments of \$1M annually with the first payment made immediately. If the prevailing annual interest rate is 3.5% compounded monthly, find the present value of the lottery's grand prize.

Solution

$$\begin{aligned} P &= \sum_{t=0}^{9} (1,000,000)\left(1+\frac{0.035}{12}\right)^{-12t} \\ &= (1,000,000)\sum_{t=0}^{9}(0.9657)^{t} \\ &= 8,587,842.94 \end{aligned}$$

Equivalence of Cash Flow Streams

The cash flow streams $\mathbf{x} = \{x_0, x_1, \ldots, x_n\}$ and $\mathbf{y} = \{y_0, y_1, \ldots, y_n\}$ are **equivalent** for an ideal bank if and only if the present values of the two streams are equal.

Example: Harvesting a Crop

Suppose you can stock a pond with fish that you can later sell for food. Stocking the pond requires an initial outlay of capital, but once stocked, the fish and pond are self-sustaining. You can choose when the harvest the fish, but the longer you wait to harvest, the larger the fish will be. The annually compounded interest rate is 5%. If you harvest after one year the cash flow stream is $\{-100, 200\}$. If you harvest after two years the cash flow stream is $\{-100, 0, 250\}$. When should you harvest?

Solution

- Harvest after one year:

$$P = -100 + \frac{200}{1 + 0.05} \approx 90.48$$

- Harvest after two years:

$$P = -100 + \frac{250}{(1 + 0.05)^2} \approx 126.76$$

You should harvest after two years.

Homework

- Read Sections 1.1–1.3
- Exercises 1–6.

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

author: J. Robert Buchanan

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