Introduction to Linear Programming

MATH 472 *Financial Mathematics*

J Robert Buchanan

2018

Objectives

In this lesson we will learn:

- the terminology of linear programming,
- explore the equivalence of several forms of linear programs,
- solve some linear programs using graphical techniques.

Linear Programming

The **Arbitrage Theorem** implies that the expected value of all possible investments is the same and equal to the risk-free-rate of return, or that it is possible to create a portfolio of investments that guarantees a return better than the risk-free rate of return for every possible outcome.

The proof of the Arbitrage Theorem requires some familiarity with linear programming.

Linear programming is a branch of mathematics concerned with optimizing a linear function of several variables subject to some set of constraints (linear equalities or inequalities) on the variables.

Example

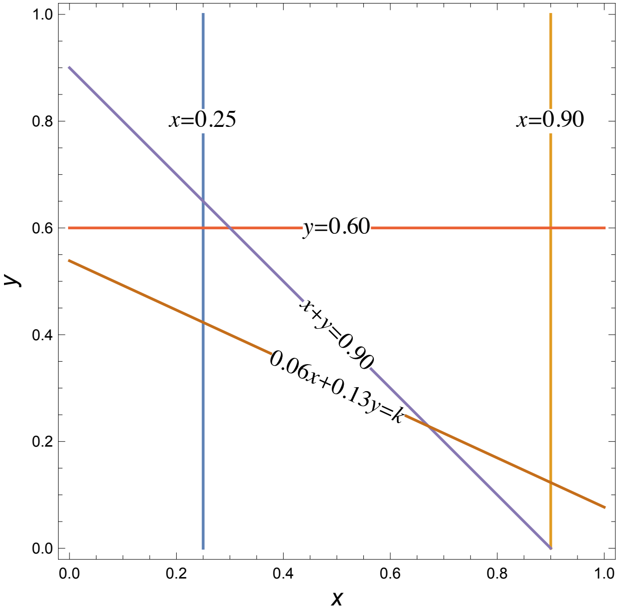
A bank may invest its deposits in loans which earn 6% interest per year and in the purchase of stocks which increase in value by 13% per year. Any un-invested amount is simply held by the bank. Suppose that government regulations require that the bank invest no more than 60% of its deposits in stocks and must keep 10% of its deposits on hand in the form of cash. As a good business practice the bank wishes to devote at least 25% of its deposits to loans. Determine how the bank should allocate its capital so as to maximize the total return on its investments.

Solution (1 of 3)

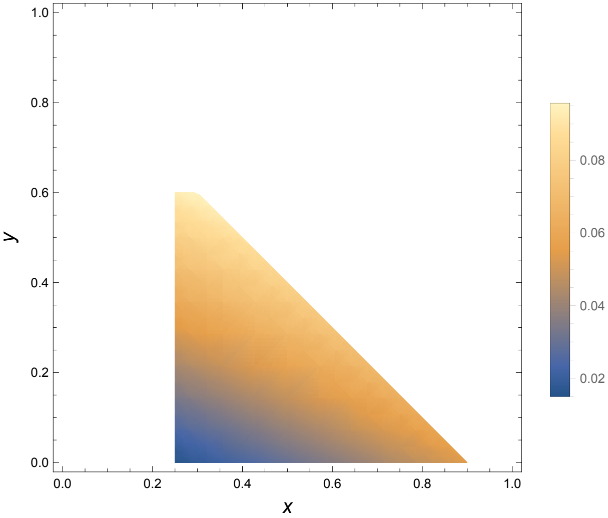
- Assume the bank can invest a fraction x in loans and fraction y in stocks.
- The total return is therefore $0.06x + 0.13y$.
- The constraints are:

$$\begin{aligned} 0.25 &\leq x \leq 0.90 \\ 0 &\leq y \leq 0.60 \\ x + y &\leq 0.90 \end{aligned}$$

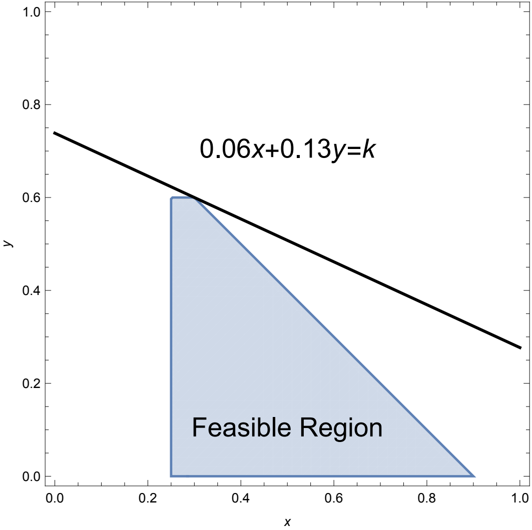
Solution (2 of 4)



Solution (3 of 4)



Solution (4 of 4)



Optimal return of $k = 0.096$ occurs when $x = 0.3$ and $y = 0.6$.

Decision Variables and Objective Functions

If **c** and **x** are vectors with n components each, the notation

$$\mathbf{c}^T\mathbf{x} = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

represents a weighted sum of the components of **x** with the weights being the components of **c**.

Remarks:

- The components of **x** are sometimes called **decision variables.**
- The weighted sum $\mathbf{c}^T\mathbf{x}$ is called an **objective function** or **cost function.**

Vector Comparisons

We write $\mathbf{u} < \mathbf{v}$ if $u_i < v_i$ for $i = 1, 2, \ldots, n$.

Similarly for

- $\mathbf{u} > \mathbf{v}$,
- $\mathbf{u} \leq \mathbf{v}$, and
- $\mathbf{u} \geq \mathbf{v}$.

Vector Comparisons

We write $\mathbf{u} < \mathbf{v}$ if $u_i < v_i$ for $i = 1, 2, \ldots, n$.

Similarly for

- $\mathbf{u} > \mathbf{v}$,
- $\mathbf{u} \leq \mathbf{v}$, and
- $\mathbf{u} \geq \mathbf{v}$.

If $\mathbf{0}$ denotes the **zero vector** then $\mathbf{x} \geq \mathbf{0}$ is an example of a **sign constraint.**

Feasible Vectors and Cost Functions

Definition

Vector $\mathbf{x}$ is **feasible** if $\mathbf{x} \geq \mathbf{0}$ and $A\mathbf{x} \leq \mathbf{b}$.

Definition

Vector $\mathbf{x}$ is an **optimal solution** if $\mathbf{x}$ is feasible and maximizes the cost function.

Constraints

Constraints on the decision variables will be expressed in the form $\mathbf{a}^T\mathbf{x} \leq z$ where $\mathbf{a}$ is a vector of n components and z is a scalar.

All relationships can be expressed using $\leq$.

$$\mathbf{a}^T\mathbf{x} \geq z \iff (-\mathbf{a})^T\mathbf{x} \leq -z$$
$$\mathbf{a}^T\mathbf{x} = z \iff \mathbf{a}^T\mathbf{x} \leq z \quad \text{and} \quad (-\mathbf{a})^T\mathbf{x} \leq -z$$

Optimization

When solving a linear program, we will

- optimize (either maximize or minimize) an objective function,
- subject to one or more constraints.

Remark: the processes of maximizing and minimizing $\mathbf{c}^T\mathbf{x}$ are equivalent in the sense that $\mathbf{c}^T\mathbf{x}$ is a maximum if and only if $(-\mathbf{c})^T\mathbf{x}$ is a minimum.

Systems of Constraints

Suppose there are m inequality constraints:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = \mathbf{a}_1^T\mathbf{x} &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = \mathbf{a}_2^T\mathbf{x} &\leq b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = \mathbf{a}_m^T\mathbf{x} &\leq b_m
\end{aligned}$$

we may express this in matrix form as

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \leq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \mathbf{b}.$$

Forms of Linear Programs

There are several equivalent forms of linear programs:

Standard form: decision variables $\mathbf{x} \geq \mathbf{0}$, constraints $A\mathbf{x} = \mathbf{b}$.

Canonical form: decision variables $\mathbf{x} \geq \mathbf{0}$, constraints $A\mathbf{x} = \mathbf{b} \geq \mathbf{0}$.

Symmetric form: decision variables $\mathbf{x} \geq \mathbf{0}$, constraints $A\mathbf{x} \leq \mathbf{b}$.

General form: constraints $A\mathbf{x} \leq \mathbf{b}$, $\hat{A}\mathbf{x} \geq \hat{\mathbf{b}}$, $\tilde{A}\mathbf{x} = \tilde{\mathbf{b}}$.

Remark: since any one of the forms can be re-cast as any of the other forms, we are free to work with the most convenient formulation in any given context.

Equivalence of Symmetric and Standard Forms

Symmetric linear program: maximize $c_1 x_1 + \cdots + c_n x_n$ subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \end{aligned}$$

Introduce **slack variables** $\hat{x}_{n+1}, \ldots, \hat{x}_{n+m}$ so that

$$\overline{\mathbf{x}} = \langle x_1, x_2, \ldots, x_n, \hat{x}_{n+1}, \hat{x}_{n+2}, \ldots, \hat{x}_{n+m} \rangle = \langle \mathbf{x}, \hat{\mathbf{x}} \rangle.$$

For $j = 1, 2, \ldots, m$ let $\hat{x}_{n+j} = b_j - \sum_{i=1}^{n} a_{ji}x_i \geq 0$.

Equivalence of Symmetric and Standard Forms

The system of constraints can now be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + \hat{x}_{n+1} &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + \hat{x}_{n+2} &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + \hat{x}_{n+m} &= b_m. \end{aligned}$$

In matrix/vector form that can be expressed as

$$\left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 1 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \\ \hline \hat{x}_{n+1} \\ \hat{x}_{n+2} \\ \vdots \\ \hat{x}_{n+m} \end{array}\right] = \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_m \end{array}\right].$$

Equivalence of Symmetric and Standard Forms

In compact form the matrix equation can be expressed as

$$\left[\begin{array}{c|c} A & I_m \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hline \hat{\mathbf{x}} \end{array}\right] = \overline{A}\,\overline{\mathbf{x}} = \mathbf{b}.$$

The objective function $\mathbf{c}^T\mathbf{x}$ can be written as

$$\begin{aligned} & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n + (0)\hat{x}_{n+1} + (0)\hat{x}_{n+2} + \cdots + (0)\hat{x}_{n+m} \\ &= \langle c_1, c_2, \ldots, c_n, 0, 0, \ldots, 0\rangle^T \langle x_1, x_2, \ldots, x_n, \hat{x}_{n+1}, \hat{x}_{n+2}, \ldots, \hat{x}_{n+m}\rangle \\ &= \langle \mathbf{c}, \mathbf{0}\rangle^T \langle \mathbf{x}, \hat{\mathbf{x}}\rangle \\ &= \overline{\mathbf{c}}^T \overline{\mathbf{x}}. \end{aligned}$$

The derived linear program: maximize $\overline{\mathbf{c}}^T\overline{\mathbf{x}}$ subject to $\overline{A}\,\overline{\mathbf{x}} = \mathbf{b}$ is of standard form.

Summary

Given the symmetric linear program: maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$, introduce **slack variables.**

1. If $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ augment with m slack variables $\hat{x}_{n+j} = b_j - \sum_{i=1}^{n} a_{ji}x_i$ for $j = 1, 2, \ldots, m$ to form decision variable:

$$\overline{\mathbf{x}} = \langle \mathbf{x}, \hat{\mathbf{x}} \rangle = \langle x_1, x_2, \ldots, x_n, \hat{x}_{n+1}, \hat{x}_{n+2}, \ldots, \hat{x}_{n+m} \rangle.$$

2. If $A \in \mathbb{R}^{m \times n}$ then augment the columns of A with the $m \times m$ identity matrix.

$$\left[\begin{array}{c|c} A & I_m \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hline \hat{\mathbf{x}} \end{array}\right] = \overline{A}\,\overline{\mathbf{x}} = \mathbf{b}$$

3. Augment $\mathbf{c}$ with m zeros, then $\overline{\mathbf{c}}^T\overline{\mathbf{x}} = \mathbf{c}^T\mathbf{x}$.

Equivalence of Standard and Symmetric Forms

Given the standard linear program: maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} = \mathbf{b}$, introduce inequality constraints.

Equivalence of Standard and Symmetric Forms

Given the standard linear program: maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} = \mathbf{b}$, introduce inequality constraints.

1. $A\mathbf{x} = \mathbf{b}$ if and only if $A\mathbf{x} \leq \mathbf{b}$ and $-A\mathbf{x} \leq -\mathbf{b}$.
2. Augment the rows of matrix A with the rows of matrix $-A$.

$$\left[\begin{array}{c} A \\ \hline -A \end{array}\right] \mathbf{x} \leq \left[\begin{array}{c} \mathbf{b} \\ \hline -\mathbf{b} \end{array}\right]$$

3. The weighted sum remains the same.

General Linear Program

The most flexible statement of a linear program relaxes the non-negativity of the decision variables and mixes the equations and inequalities of the constraints.

A linear program of the form: maximize $\mathbf{c}^T\mathbf{x}$ subject to

$$\begin{aligned} A\mathbf{x} &\leq \mathbf{b} \\ \hat{A}\mathbf{x} &\geq \hat{\mathbf{b}} \\ \tilde{A}\mathbf{x} &= \tilde{\mathbf{b}} \end{aligned}$$

is called a **general linear program.**

General Linear Program

The most flexible statement of a linear program relaxes the non-negativity of the decision variables and mixes the equations and inequalities of the constraints.

A linear program of the form: maximize $\mathbf{c}^T\mathbf{x}$ subject to

$$\begin{aligned} A\mathbf{x} &\leq \mathbf{b} \\ \hat{A}\mathbf{x} &\geq \hat{\mathbf{b}} \\ \tilde{A}\mathbf{x} &= \tilde{\mathbf{b}} \end{aligned}$$

is called a **general linear program.**

Remark: every standard, canonical, or symmetric linear program is trivially a general linear program. The converse is also true.

Positive and Negative Parts

Definition

The **positive part** of real number x is denoted x^+ and is

$$x^+ = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

The **negative part** of real number x is denoted x^- and is

$$x^- = \begin{cases} -x & \text{if } x \leq 0, \\ 0 & \text{if } x > 0. \end{cases}$$

Remark: this definition can be applied component-wise to real vectors.

Equivalence of General and Symmetric Programs

Given a linear program in general form, we can construct a symmetric linear program.

1. If the decision vector $\mathbf{x} \in \mathbb{R}^n$ is unrestricted in sign, create a new decision vector $\langle \mathbf{x}^+, \mathbf{x}^- \rangle \in \mathbb{R}^{2n}$.

$$\langle \mathbf{x}^+, \mathbf{x}^- \rangle \geq \mathbf{0}$$

2. Create a new vector of weights $\langle \mathbf{c}, -\mathbf{c} \rangle$.

$$\langle \mathbf{c}, -\mathbf{c} \rangle^T \langle \mathbf{x}^+, \mathbf{x}^- \rangle = \mathbf{c}^T (\mathbf{x}^+ - \mathbf{x}^-) = \mathbf{c}^T \mathbf{x}$$

Equivalence of General and Symmetric Programs

3. The system of constraints is re-written in inequality form:

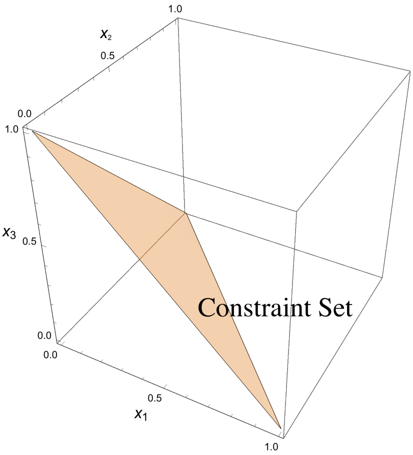
$$\begin{bmatrix} A & -A \\ -\hat{A} & \hat{A} \\ \tilde{A} & -\tilde{A} \\ -\tilde{A} & \tilde{A} \end{bmatrix} \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ -\hat{\mathbf{b}} \\ \tilde{\mathbf{b}} \\ -\tilde{\mathbf{b}} \end{bmatrix}.$$

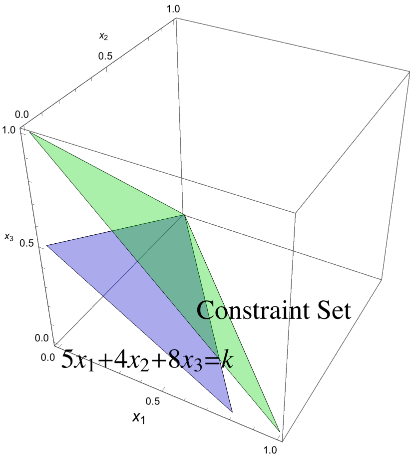
Example

Use the notion of the intersection of planes in $\mathbb{R}^3$ to minimize $\mathbf{c}^T\mathbf{x} = 5x_1 + 4x_2 + 8x_3$ subject to $A\mathbf{x} = x_1 + x_2 + x_3 = 1$ and $\mathbf{x}$ is feasible.

Remark: this is a linear program stated in standard form.

Graphical Solution





The cost function has a minimum of 4 at $\mathbf{x} = \langle 0, 1, 0 \rangle$.

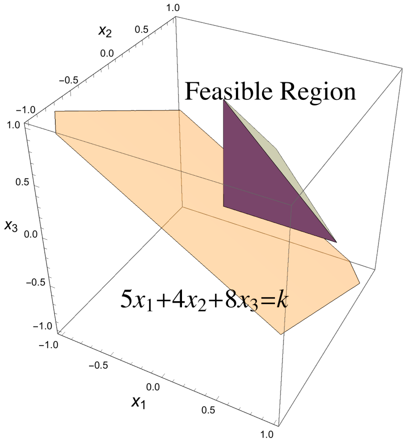
Example

Use the notion of the intersection of planes in $\mathbb{R}^3$ to minimize $\mathbf{c}^T\mathbf{x} = 5x_1 + 4x_2 + 8x_3$ subject to $A\mathbf{x} = x_1 + x_2 + x_3 \leq 1$ and $\mathbf{x}$ is feasible.

Remark: this linear program is stated in symmetric form.

Graphical Solution

If the constraints are $x_1 + x_2 + x_3 \leq 1$ and $\mathbf{x}$ feasible, then the set of points where the solution must be found would resemble a tetrahedron with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.



The cost function has a minimum of 0 at $\mathbf{x} = \langle 0, 0, 0 \rangle$.

Homework

- Read Section
- Exercises:

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

author: J. Robert Buchanan

publisher: World Scientific Publishing Co. Pte. Ltd.

address: 27 Warren St., Suite 401–402, Hackensack, NJ 07601

ISBN: 978-9814407441