# Lognormal Random Variables and Probability

MATH 472 *Financial Mathematics*

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# Objectives

In this lesson we will:

- define the lognormal probability distribution,
- discuss some of the properties of lognormal random variables,
- apply lognormal random variables to model the movements of equity prices.

# Lognormal Random Variables

**Definition**

A random variable $X$ is a **lognormal random variable** with parameters $\mu$ and $\sigma$ if $\ln X$ is a normally distributed random variable with mean $\mu$ and variance $\sigma^2$.

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**Remarks:**

- The parameter $\mu$ is sometimes called the **drift.**
- The parameter $\sigma$ is sometimes called the **volatility**.

# Lognormal PDF (1 of 2)

Suppose $X$ is lognormal with parameters $\mu$ and $\sigma$, then $Y = \ln X$ is normal with mean $\mu$ and variance $\sigma^2$.

$$\begin{aligned}\mathbb{P}(X < x) &= \mathbb{P}(Y < \ln x) = \Phi(\ln x) \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\ln x} e^{-(t-\mu)^2/2\sigma^2}\, dt.\end{aligned}$$

If we let $u = e^t$ and $du = e^t\, dt$, then

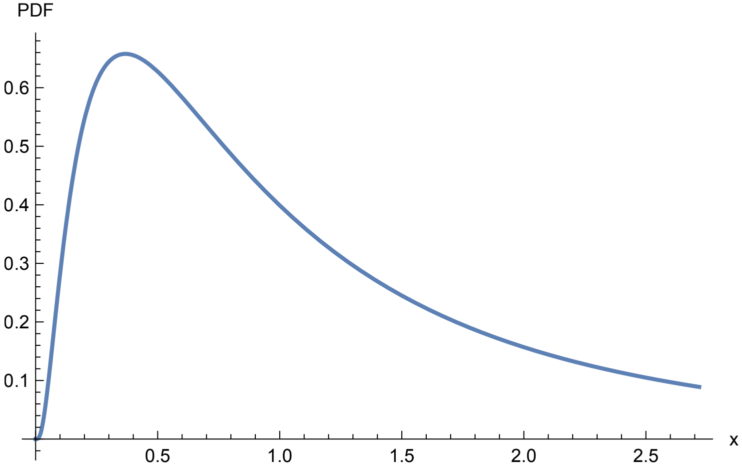
$$\mathbb{P}(X < x) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^x \frac{1}{u} e^{-(\ln u-\mu)^2/2\sigma^2}\, du,$$

the cumulative distribution function for the lognormally distributed random variable $X$.

# Lognormal PDF (2 of 2)

The probability density function for lognormal $X$ is

$$f(x) = \frac{1}{(\sigma\sqrt{2\pi})x} e^{-(\ln x - \mu)^2/2\sigma^2}.$$



# Mean and Variance of a Lognormal RV

**Lemma**

*If $X$ is a lognormal random variable with parameters $\mu$ and $\sigma$ then*

$$\begin{aligned}\mathbb{E}\left(X\right) &= e^{\mu+\sigma^2/2},\\ \mathrm{Var}\left(X\right) &= e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right).\end{aligned}$$

# Proof (1 of 2)

Let $X$ be lognormally distributed with parameters $\mu$ and $\sigma$, then

$$\begin{aligned}
\mathbb{E}(X) &= \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty x \left( \frac{1}{x} e^{-(\ln x - \mu)^2/2\sigma^2} \right) dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty e^t e^{-(t-\mu)^2/2\sigma^2} \, dt \\
&= e^{\mu+\sigma^2/2} \left[ \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty e^{-(t-(\mu+\sigma^2))^2/2\sigma^2} \, dt \right] \\
&= e^{\mu+\sigma^2/2}.
\end{aligned}$$

# Proof (2 of 2)

Let $X$ be lognormally distributed with parameters $\mu$ and $\sigma$, then

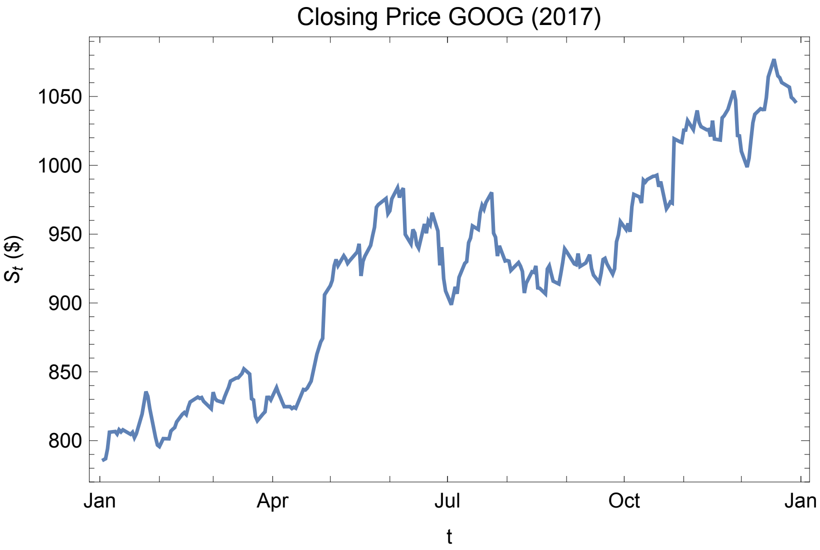
$$
\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}\left(X^2\right) - (\mathbb{E}(X))^2 \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty x^2 \left(\frac{1}{x} e^{-(\ln x - \mu)^2/2\sigma^2}\right) dx - \left(e^{\mu+\sigma^2/2}\right)^2 \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty e^{2t} e^{-(t-\mu)^2/2\sigma^2} dt - e^{2\mu+\sigma^2} \\
&= e^{2(\mu+\sigma^2)} \left[\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty e^{-(t-(\mu+2\sigma))^2/2\sigma^2} dt\right] - e^{2\mu+\sigma^2} \\
&= e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2} \\
&= e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right).
\end{aligned}
$$

# Lognormal RVs and Security Prices

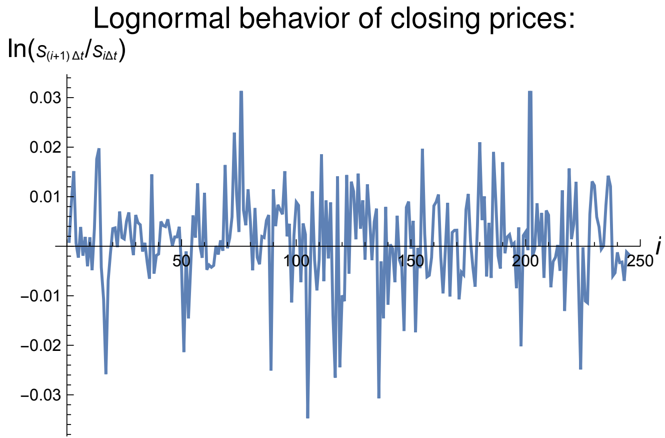
**Observation:**

- Let $S_0$ denote the price of a security at some starting time arbitrarily chosen to be $t = 0$.
- For $t \geq 1$, let $S_t$ denote the price of the security on day $t$.
- The random variable $X_t = \dfrac{S_t}{S_{t-1}}$ for $t \geq 1$ is lognormally distributed, *i.e.*, $\ln X_t = \ln S_t - \ln S_{t-1}$ is normally distributed.

# Closing Prices of Google (GOOG) Stock

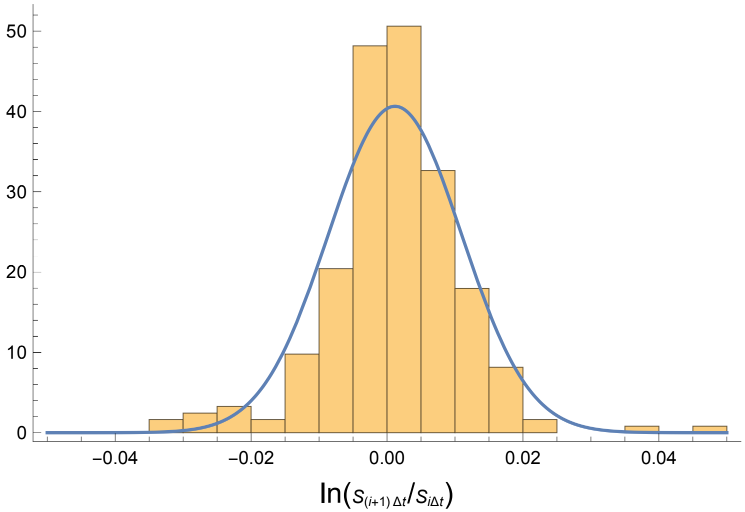


# Lognormal Behavior of Sony (SNE) Stock




$$\mu = 0.00116725 \quad \sigma = 0.00981725$$

# Histogram of Lognormal Behavior



# Example (1 of 2)

What is the probability that the closing price of Google stock will be higher today than yesterday?

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$$\begin{aligned}
\mathbb{P}\left(\underbrace{\frac{S_t}{S_{t-1}}}_{\text{lognormal}} > 1\right) &= \mathbb{P}\left(\underbrace{\ln \frac{S_t}{S_{t-1}}}_{\text{normal}} > \ln 1\right) \\
&= \mathbb{P}(X > 0) \\
&= \mathbb{P}\left(Z > \frac{0 - 0.00116725}{0.00981725}\right) \\
&= 1 - \mathbb{P}(Z \leq -0.118898) \\
&= 1 - \Phi(-0.118898) \\
&= 0.547322
\end{aligned}$$

# Example (2 of 2)

What is the probability that tomorrow's closing price will be higher than yesterday's closing price?

# Example (2 of 2)

What is the probability that tomorrow's closing price will be higher than yesterday's closing price?

$$\begin{aligned}
\mathbb{P}\left(\frac{S_{t+1}}{S_{t-1}} > 1\right) &= \mathbb{P}\left(\frac{S_{t+1}}{S_t}\frac{S_t}{S_{t-1}} > 1\right) \\
&= \mathbb{P}\left(\ln\frac{S_{t+1}}{S_t} + \ln\frac{S_t}{S_{t-1}} > 0\right) \\
&= \mathbb{P}(X + X > 0) \\
&= \mathbb{P}\left(Z > \frac{0 - 2(0.00116725)}{\sqrt{2(0.0091725)^2}}\right) \\
&= 1 - \mathbb{P}(Z \leq -0.168147) \\
&= 1 - \Phi(-0.168147) \\
&= 0.566766
\end{aligned}$$

# Properties of Expected Value and Variance

If an item is worth $K$ but can only be sold for $X$, a rational investor would sell only if $X \geq K$.

The net **payoff** of the sale can be expressed as

$$(X - K)^+ = \begin{cases} X - K & \text{if } X \geq K, \\ 0 & \text{if } X < K. \end{cases}$$

# Payoff When $X$ is Normal

## Corollary

*If $X$ is normal random variable with mean $\mu$ and variance $\sigma^2$ and $K$ is a constant, then*

$$\mathbb{E}\left((X-K)^+\right) = \frac{\sigma}{\sqrt{2\pi}} e^{-(\mu-K)^2/2\sigma^2} + (\mu-K)\Phi\left(\frac{\mu-K}{\sigma}\right),$$

$$\begin{aligned}
&\operatorname{Var}\left((X-K)^+\right) \\
&= \left((\mu-K)^2+\sigma^2\right)\Phi\left(\frac{\mu-K}{\sigma}\right) + \frac{(\mu-K)\sigma}{\sqrt{2\pi}} e^{-(\mu-K)^2/2\sigma^2} \\
&\quad - \left(\frac{\sigma}{\sqrt{2\pi}} e^{-(\mu-K)^2/2\sigma^2} + (\mu-K)\Phi\left(\frac{\mu-K}{\sigma}\right)\right)^2.
\end{aligned}$$

# Payoff When $X$ is Lognormal

## Corollary

*If $X$ is a lognormally distributed random variable with parameters $\mu$ and $\sigma^2$ and $K > 0$ is a constant then*

$$\begin{aligned}
\mathbb{E}\left((X-K)^+\right) &= e^{\mu+\sigma^2/2}\Phi\left(\frac{\mu-\ln K}{\sigma}+\sigma\right) - K\Phi\left(\frac{\mu-\ln K}{\sigma}\right), \\
\mathrm{Var}\left((X-K)^+\right) &= e^{2(\mu+\sigma^2)}\Phi\left(w+2\sigma\right) + K^2\Phi\left(w\right) \\
&\quad - 2Ke^{\mu+\sigma^2/2}\Phi\left(w+\sigma\right) \\
&\quad - \left(e^{\mu+\sigma^2/2}\Phi\left(w+\sigma\right) - K\Phi\left(w\right)\right)^2
\end{aligned}$$

*where $w = (\mu - \ln K)/\sigma$.*

# Homework

- Read Section 3.7
- Exercises: 19–23, 25–29

# Credits

These slides are adapted from the textbook,


*An Undergraduate Introduction to Financial Mathematics*, 3rd edition, (2012).

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