Options
An Undergraduate Introduction to Financial Mathematics

J. Robert Buchanan

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Definition
An **option** is the right, but *not* the obligation, to buy or sell a security such as a stock for an agreed upon price at some time in the future.
Definitions and Terminology

Definition
An **option** is the right, but *not* the obligation, to buy or sell a security such as a stock for an agreed upon price at some time in the future.

**strike price**: agreed upon price for buying or selling.

**expiry**: deadline by which the option must be exercised (also known as **exercise time**, **strike time**, and **expiry date**).

**call option**: an option to buy a security (sometimes just called a **call**).

**put option**: an option to sell a security (a **put** for short).
Some Types of Options

**European option**: can only be exercised at expiry.

**American option**: can be exercised at or before expiry.
Some Types of Options

**European option:** can only be exercised at expiry.

**American option:** can be exercised at or before expiry.

Other types exist such as *Asian, Bermudan, look-back, etc.*

- Our objective is to determine a method for pricing the European-style options.
- Their values satisfy the **Black-Scholes partial differential equation**.
Notation

\( C^a \): value of an American-style call option
\( C^e \): value of a European-style call option
\( K \): strike price of an option
\( P^a \): value of an American-style put option
\( P^e \): value of a European-style put option
\( r \): continuously compounded, risk-free interest rate
\( \delta \): continuously compounded, dividend yield rate
\( S \): price of a share of a security
\( T \): exercise time or expiry of an option (sometimes called the strike time)
\( t \): current time, generally with \( 0 \leq t \leq T \)
Payoff for a Call Option

A call option does not exhibit a positive payoff until the security price exceeds the strike price.

\[(S(T) - K)^+\]
Profit for a Call Option

The payoff of a call option minus its cost is the call’s profit.

\[(S(T) - K)^+ - C\]
Payoff for a Put Option

A put option does not exhibit a positive payoff until the strike price exceeds the security price.

\[(K - S(T))^+\]
Profit for a Put Option

The payoff of a put option minus its cost is the put’s profit.

\[(K - S(T))^+ - P\]
Theorem
\[ C^a \geq C^e \text{ and } P^a \geq P^e. \]

An American form of an option will always be worth as much as the European version of the option (all other features being the same).
Assume: $C^a < C^e$.

- Sell the European option for $C^e$ and purchase the American option for $C^a$.
- This generates cash flow $C^e - C^a > 0$ at time $t = 0$ which is invested at the risk-free rate $r$.
- If the European option holder chooses to exercise the option at expiry, the American option holder can also exercise.
- If the European option holder does not exercise, the American option holder can let the American option expire unused.
- At expiry the seller still holds $(C^e - C^a)e^{rT} > 0$. 
Properties of Options (3 of 4)

Theorem
\[ C^e \geq S - Ke^{-rT}. \]
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Remark: this is equivalent to the inequality
\[ C^e + Ke^{-rT} \geq S. \]

It can be interpreted as stating that the cost of a European call plus the present call of the strike is always at least the cost of the security.
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Develop a similar inequality relating a put, the strike price, and the security.
Properties of Options (4 of 4)

Assume: $C^e < S - Ke^{-rT}$.

- Short the security for $S$ and purchase the European call for $C^e$.
- This generates cash flow $S - C^e$ at time $t = 0$. This is invested at the continuously compounded risk-free rate $r$.
- At expiry the option holder has a risk-free investment worth $(S - C^e)e^{rT}$ and spends no more than $K$ to close out the short position in the security.
- At expiry the option holder has $(S - C^e)e^{rT} - K > 0$ since this inequality is equivalent to the assumption that $C^e < S - Ke^{-rT}$.
- Therefore a risk-free positive profit can be had.
Theorem

For non-dividend paying stocks, if the European put and call have the same strike price and expiry, then

\[ P^e + S = C^e + Ke^{-rT}. \]
Put-Call Parity Formula

Theorem
For non-dividend paying stocks, if the European put and call have the same strike price and expiry, then

\[ P^e + S = C^e + Ke^{-rT}. \]

Put-Call Parity can be interpreted as stating that the cost of a European put plus the security equals the cost of the European call plus the present value of the strike.
Proof (1 of 2)

**Assume:** $P^e + S < C^e + Ke^{-rT}$.

- Borrow $P^e + S - C^e$ at the continuously compounded risk-free rate $r$.
- Purchase the European put option for $P^e$, the security for $S$, and sell the European call option for $C^e$.
- At expiry sell the security for at least $K$ and pay back the loan with interest.
- At expiry this leaves the borrower with $K - (P^e + S - C^e)e^{rT} > 0$ since this inequality is equivalent to the assumption that $P^e + S < C^e + Ke^{-rT}$.
- Therefore a risk-free positive profit can be had.
Proof (2 of 2)

Assume: $P^e + S > C^e + Ke^{-rT}$.

- Short the security for $S$, sell the European put option for $P^e$, and purchase the European call option for $C^e$.

- At time $t = 0$ this generates a cash flow of $S + P^e - C^e > 0$. Invest this amount at the continuously compounded risk-free rate $r$.

- At expiry purchase the security for at most $K$ and close out the short position.

- At expiry the short seller holds $(S + P^e - C^e)e^{rT} - K > 0$ since this inequality is equivalent to the assumption that $P^e + S > C^e + Ke^{-rT}$.
Effect of Dividends

- Suppose a corporation will pay a dividend to the shareholders at time $t_d$.
- The amount of the dividend will be $\delta S(t_d)$. 
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- Consider the one-sided limits:

$$\lim_{t \to t_d^-} S(t) = S(t_d^-)$$

$$\lim_{t \to t_d^+} S(t) = S(t_d^+)$$

These are the values of the corporation’s stock just before and just after the dividend is paid.
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  These are the values of the corporation’s stock just before and just after the dividend is paid.
- In the absence of arbitrage,
  \[
  S(t_d^+) = (1 - \delta)S(t_d^-).
  \]
Theorem
If $n$ dividend payments of the form $\delta S(t_i^-)$ will be made at times $t_i^-$ for $i = 1, 2, \ldots, n$ then the Put-Call Parity Formula for discrete dividend payments can be expressed as

$$P^e + S(0) - \delta \sum_{i=1}^{n} S(t_i^-) e^{-rt_i} = C^e + Ke^{-rT}.$$
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P^e + S(0) - \delta \sum_{i=1}^{n} S(t_i^-) e^{-rt_i} = C^e + Ke^{-rT}.
\]

The value of the security is discounted by the total of the present values of the dividends paid.
Continuous Dividends

Theorem

For European options on securities which pay dividends at a continuous, constant dividend yield $\delta$, the Put-Call Parity Formula takes on the form

$$P^e + S(0)e^{-\delta T} = C^e + Ke^{-rT}.$$
Call options can be thought of as insurance against a rise in the price of a security.

An investor does not know with certainty the value of the security at the strike time, so how much should be paid for the call?

Example

Suppose $S(0) = \$100$, at $t = T$.

$S(T) = \begin{cases} 
\$200 & \text{with probability } p, \\
\$50 & \text{with probability } 1 - p. 
\end{cases}$

An investor can purchase a European call option whose value is $C$. The exercise time and strike price of the option are respectively $T = 1$ and $\$150$. In the absence of arbitrage what is the value of $C$?
Binary Model

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An investor can purchase a European call option whose value is $C$. The exercise time and strike price of the option are respectively $T = 1$ and $150$. In the absence of arbitrage what is the value of $C$?
Present Value, Arbitrage-free Setting

- Since two distinct times \((t = 0 \text{ and } t = 1)\) are involved we must find the present values of all quantities being compared.
- Assume the simple interest rate for the interval \(0 \leq t \leq 1\) is \(r\).
- In the absence of arbitrage, there should be no expected profit from either purchasing the security or the call option.
Purchase the Stock

Suppose an investor purchases the security.

- Initial cash flow: $-100.
- At time $t = 1$, the payoff is $\text{payoff} = \begin{cases} -100 + \frac{200}{1 + r} & \text{with probability } p, \\ -100 + \frac{50}{1 + r} & \text{with probability } 1 - p. \end{cases}

If the expected value of the payoff is 0, then

$$0 = p \left[-100 + \frac{200}{1 + r}\right] + (1 - p) \left[-100 + \frac{50}{1 + r}\right]$$

$$p = \frac{2}{3}r + \frac{1}{3}.$$
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  \end{cases}
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- If the expected value of the payoff is 0, then

  \[
  0 = p \left( -100 + \frac{200}{1 + r} \right) + (1 - p) \left( -100 + \frac{50}{1 + r} \right)
  \]

  \[
  p = \frac{2r + 1}{3}.
  \]
Purchase the Option

Suppose the investor purchases the call option.

Initial cash flow: $-C$.

At $t=T$, the payoff is:

\[\text{payoff} = \begin{cases} 
-C + \left(\frac{200 - 150}{1 + r}\right) & \text{with probability } p, \\
-C + \left(\frac{50 - 150}{1 + r}\right) + \left(\frac{1}{1 + r}\right) & \text{with probability } 1 - p.
\end{cases}\]

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\[C = 50 \left(\frac{2}{3} r + \frac{1}{3}\right).\]
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Purchase the Option

Suppose the investor purchases the call option.

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- At $t = T$, the payoff is

$$\text{payoff} = \begin{cases} 
-C + (\$200 - \$150)/(1 + r) & \text{with probability } p, \\
-C + (\$50 - \$150)^+/ (1 + r) & \text{with probability } 1 - p.
\end{cases}$$
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Suppose the investor purchases the call option.

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  -C + (50 - 150)/(1 + r) & \text{with probability } 1 - p.
\end{cases}
\]

- If the expected value of the payoff is 0, then

\[
0 = p \left[ -C + \frac{200 - 150}{1 + r} \right] + (1 - p) \left[ -C + \frac{50 - 150}{1 + r} \right] \\
C = \frac{50(2r + 1)}{3(r + 1)}.
\]
Parametric Plot of \((p(r), C(r))\)

If \((p(r), C(r)) = \left( \frac{2r + 1}{3}, \frac{50(2r + 1)}{3(r + 1)} \right)\) then as the probability of the security reaching $200 at \(T = 1\) increases, so does the cost of the option.
Example (1 of 3)

- Suppose the current value of the stock is \( S(0) = $100 \) and at time \( T = 1 \)

\[
S(1) = \begin{cases} 
$150 & \text{with probability } p = 0.45, \\
$75 & \text{with probability } 1 - p = 0.55. 
\end{cases}
\]

- Suppose further that the risk-free interest rate is \( r = 8.388\% \).

- A European call option with a strike price of $125 can be purchased for $10 (note the arbitrage-free price is $10.3448).
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- Suppose further that the risk-free interest rate is $r = 8.388\%$.

- A European call option with a strike price of $125$ can be purchased for $10$ (note the arbitrage-free price is $10.3448$).

- Design an investment scheme which guarantees a positive profit.
Solution: Borrow funds to take a position of $x$ shares of the stock and $y$ call options ($x$ and $y$ can be positive or negative).

- At $t = 0$, the portfolio is worth $100x + 10y$.
- At $t = T$, the investor owes $(100x + 10y)e^{0.08338}$.
- If $S(1) = 150$, the portfolio generates a cash flow of
  \[ 150x + (150 - 125)y = 150x + 25y. \]
- If $S(1) = 75$, the portfolio generates a cash flow of $75x$. 
Example (3 of 3)

A positive profit is guaranteed in the region where

\[ 150x + 25y > (100x + 10y)e^{0.08338} \]
\[ 75x > (100x + 10y)e^{0.08338} \]

which implies \( 3x + y > 0 \).
Black-Scholes Equation

We now turn our attention to mathematically modeling the value of a European call option.

- Suppose a stock obeys an Itô process of the form:
  \[ dS = \mu S \, dt + \sigma S \, dW(t) \]

- An investor will create a portfolio \( Y \), consisting of a short position in a European call option and a long position of \( \Delta \) shares of the stock.
  \[ Y = F(S, t) = C^e(S, t) - (\Delta)S \]
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  \[ Y = F(S, t) = C^e(S, t) - (\Delta)S \]

- Use Itô’s lemma to find the stochastic process followed by \( Y \).
Itô Process for $dY$

$$
\begin{align*}
    dY &= \left( \mu SY_S + \frac{1}{2} \sigma^2 S^2 Y_{SS} + Y_t \right) \, dt + (\sigma SY_S) \, dW(t) \\
    &= \left( \mu S \left[ \frac{\partial C^e}{\partial S} - \Delta \right] + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C^e}{\partial S^2} + \frac{\partial C^e}{\partial t} \right) \, dt \\
    &\quad + \sigma S \left( \frac{\partial C^e}{\partial S} - \Delta \right) \, dW(t)
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Note: the process becomes deterministic if $\Delta = \frac{\partial C^e}{\partial S}$.
Itô Process for \( dY \)

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    dY = \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C^e}{\partial S^2} + \frac{\partial C^e}{\partial t} \right) \, dt
\]
No Arbitrage Assumption

The payoff from the portfolio should be the same as that generated by investing an equivalent amount of money to \( Y \) in savings earning interest compounded continuously at rate \( r \).

\[
dY = rY \, dt \\
= r(C^e - (\Delta)S) \, dt \\
= r \left( C^e - S \frac{\partial C^e}{\partial S} \right) \, dt
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Recall from Itô’s lemma that

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\]

Equating the two expressions for $dY$ yields the **Black-Scholes partial differential equation**

\[
   r \, C^e = \frac{\partial C^e}{\partial t} + r \, S \frac{\partial C^e}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C^e}{\partial S^2}.
\]
In order to solve the Black-Scholes PDE we must have some boundary and final conditions.

At expiry the call option is worth \((S(T) - K)\) + , so \(C_e(S(T), T) = (S(T) - K)\) + , this is the final condition.

The stock will have a value in the interval \([0, \infty)\). The boundary at \(S = 0\) is absorbing, so if there is a time \(t^* \geq 0\) such that \(S(t^*) = 0\), then \(S(t) = 0\) for all \(t \geq t^*\). In this case the option will never be exercised and is worthless. Thus \(C_e(0, t) = 0\), which is the boundary condition at \(S = 0\).
Final and Boundary Conditions (1 of 2)

In order to solve the Black-Scholes PDE we must have some boundary and final conditions.

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$$C^e(0, t) = 0,$$

which is the boundary condition at $S = 0$. 
From the Put-Call Parity Formula:

\[ C^e = P^e + S - Ke^{-rT} \]

\[ \lim_{S \to \infty} C^e = \lim_{S \to \infty} P^e + S - Ke^{-rT} \]

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As the security unbounded in value:

- a put option (right to sell at a finite price) becomes worthless, and
- the call option is worth the difference between the security price and the present value of the strike.
Initial Boundary Value Problem

For \((S, t)\) in \([0, \infty) \times [0, T]\),

\[
\begin{align*}
    rC^e &= \frac{\partial C^e}{\partial t} + rS \frac{\partial C^e}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C^e}{\partial S^2}, \\
    C^e(S, T) &= (S(T) - K)^+ \quad \text{for } S > 0, \\
    C^e(0, t) &= 0 \quad \text{for } 0 \leq t < T, \\
    C^e(S, t) &= S - Ke^{-r(T-t)} \quad \text{as } S \to \infty.
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The solution of this IBVP is the subject of the next chapter.
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Spread

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Definition
A long call position with a strike price of $K_1$ and a short call position with a strike price of $K_2 > K_1$ is called a **bull spread**.

Remark: We are assuming the underlying stock is the same and the expiry of the two calls is the same.
Bull Spread (1 of 2)

- Since $K_2 > K_1$ then $C_2 < C_1$ (why?).
- Initial outlay of capital in amount $C_1 - C_2 > 0$.
- Payoff of long call $(S(T) - K_1)^+$. 
- Payoff of short call $-(S(T) - K_2)^+$. 

<table>
<thead>
<tr>
<th></th>
<th>Long Call</th>
<th>Short Call</th>
<th>Total Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(T)$</td>
<td>$K_1$</td>
<td>$0$</td>
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<td>$K_1 &lt; S(T) &lt; K_2$</td>
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<td>$0$</td>
<td>$(S(T) - K_1)^+$</td>
</tr>
<tr>
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Initial outlay of capital in amount \( C_1 - C_2 > 0 \).

Payoff of long call \((S(T) - K_1)^+\).

Payoff of short call \(-(S(T) - K_2)^+\).

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<td>( K_2 - S(T) )</td>
<td>( K_2 - K_1 )</td>
</tr>
</tbody>
</table>
Bull Spread (2 of 2)
Example: Bull Spread

Example
Suppose we create a bull spread purchasing a call option with strike price $115 and selling a call option with strike price $130. Suppose that $C(115) = 5$ and $C(130) = 3$. Find the payoff and net profit if at expiry,

- $S = 110$
- $S = 115$
- $S = 125$
- $S = 130$
- $S = 135$. 
Bull Spread with Puts (1 of 2)

Suppose an investor has a long position in a put with strike price $K_1$ and a short position in a put with strike price $K_2 > K_1$.

<table>
<thead>
<tr>
<th>$S(T)$</th>
<th>Long Put Payoff</th>
<th>Short Put Payoff</th>
<th>Total Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(T) \leq K_1$</td>
<td>$K_1 - S(T)$</td>
<td>$S(T) - K_2$</td>
<td>$K_1 - K_2$</td>
</tr>
<tr>
<td>$K_1 &lt; S(T) &lt; K_2$</td>
<td>$0$</td>
<td>$S(T) - K_2$</td>
<td>$S(T) - K_2$</td>
</tr>
<tr>
<td>$K_2 \leq S(T)$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
Bull Spread with Puts (2 of 2)

Payoff/Profit

$P_2 - P_1$

$K_1$

$K_2$

$S(T)$
Example: Bull Spread

Example
Suppose we create a bull spread purchasing a put option with strike price $95 and selling a put option with strike price $105. Suppose that $P(95) = 5$ and $P(105) = 8$. Find the payoff and net profit if at expiry,

- $S = 90$
- $S = 95$
- $S = 100$
- $S = 105$
- $S = 110$. 
Bear Spreads

Definition
A short call position with a strike price of $K_1$ and a long call position with a strike price of $K_2 > K_1$ is called a bear spread.

Remarks:
- We are assuming the underlying stock is the same and the expiry of the two calls is the same.
- The positions in the bear spread are opposite those of the bull spread.
Bear Spread (1 of 2)

- Since $K_2 > K_1$ then $C_2 < C_1$.
- Initial income of capital in amount $C_1 - C_2 > 0$.
- Payoff of long call $(S(T) - K_2)^+$. 
- Payoff of short call $-(S(T) - K_1)^+$. 

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
& Long Call & Short Call & Total Payoff \\
\hline
Payoff & $S(T)$ & $K_1$ & $K_1 - S(T)$ \\
\hline
Payoff & $K_1$ & $(S(T) - K_1)^+$ & $K_1$ \\
\hline
Payoff & $S(T)$ & $K_2$ & $K_2 - S(T)$ \\
\hline
\end{tabular}
\end{center}
Since $K_2 > K_1$ then $C_2 < C_1$.

Initial income of capital in amount $C_1 - C_2 > 0$.

Payoff of long call $(S(T) - K_2)^+.$

Payoff of short call $-(S(T) - K_1)^+.$

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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$K_1 &lt; S(T) &lt; K_2$</td>
<td>0</td>
<td>$K_1 - S(T)$</td>
<td>$K_1 - S(T)$</td>
</tr>
<tr>
<td>$K_2 \leq S(T)$</td>
<td>$S(T) - K_2$</td>
<td>$K_1 - S(T)$</td>
<td>$K_1 - K_2$</td>
</tr>
</tbody>
</table>
Bear Spread (2 of 2)

Payoff/Profit

$C_1 - C_2$

$S(T)$

$K_1$

$K_2$

$K_1$ to $K_2$
Example: Bear Spread

Example
Suppose we create a bear spread purchasing a call option with strike price $150 and selling a call option with strike price $125. Suppose that $C(150) = 5$ and $C(125) = 10$. Find the payoff and net profit if at expiry,

- $S = 120$
- $S = 125$
- $S = 150$
- $S = 150$
- $S = 160$. 
Suppose an investor has a short position in a put with strike price $K_1$ and a long position in a put with strike price $K_2 > K_1$.

<table>
<thead>
<tr>
<th>$S(T)$</th>
<th>Long Put Payoff</th>
<th>Short Put Payoff</th>
<th>Total Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(T) \leq K_1$</td>
<td>$K_2 - S(T)$</td>
<td>$S(T) - K_1$</td>
<td>$K_2 - K_1$</td>
</tr>
<tr>
<td>$K_1 &lt; S(T) &lt; K_2$</td>
<td>$K_2 - S(T)$</td>
<td>0</td>
<td>$K_2 - S(T)$</td>
</tr>
<tr>
<td>$K_2 \leq S(T)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Bear Spread with Puts (2 of 2)

Payoff/Profit

\[ P_2 - P_1 \]

\[ S(T) \]

\[ K_1 \]

\[ K_2 \]
Example: Bear Spread

Example
Suppose we create a bear spread selling a put option with strike price $50 and purchasing a put option with strike price $60. Suppose that $P(50) = 3$ and $P(60) = 5$. Find the payoff and net profit if at expiry,

- $S = 45$
- $S = 50$
- $S = 55$
- $S = 60$
- $S = 65$. 
Butterfly Spreads

Definition
A long call position with a strike price of $K_1$, a long call position with a strike price of $K_3 > K_1$, and a short position in two calls with strike price $K_2 = (K_1 + K_3)/2$ is called a butterfly spread.

Remark: We are assuming the underlying stock is the same and the expiry of the three calls is the same.
Butterfly Spread

- Note that $K_3 > K_2 > K_1$.
- Payoffs of long calls are $(S(T) - K_1)^+$ and $(S(T) - K_3)^+$.
- Payoff of two short calls $-2(S(T) - K_2)^+$. 

<table>
<thead>
<tr>
<th>Payoff 1st</th>
<th>Payoff 2nd</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(T)$</td>
<td>$S(T)$</td>
<td>$S(T)$</td>
</tr>
<tr>
<td>$K_1$</td>
<td>$S(T)$</td>
<td>$S(T)$</td>
</tr>
<tr>
<td>$S(T)$</td>
<td>$K_3$</td>
<td>$S(T)$</td>
</tr>
<tr>
<td>$K_2$</td>
<td>$S(T)$</td>
<td>$S(T)$</td>
</tr>
<tr>
<td>$S(T)$</td>
<td>$K_3$</td>
<td>$S(T)$</td>
</tr>
</tbody>
</table>
Butterfly Spread

- Note that $K_3 > K_2 > K_1$.
- Payoffs of long calls are $(S(T) - K_1)^+$ and $(S(T) - K_3)^+$.
- Payoff of two short calls $-2(S(T) - K_2)^+$.

<table>
<thead>
<tr>
<th>$S(T)$</th>
<th>Payoff 1st Long Call</th>
<th>Payoff 2nd Long Call</th>
<th>Payoff Short Calls</th>
<th>Payoff Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(T) \leq K_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$K_1 &lt; S(T) &lt; K_2$</td>
<td>$S(T) - K_1$</td>
<td>0</td>
<td>0</td>
<td>$S(T) - K_1$</td>
</tr>
<tr>
<td>$K_2 \leq S(T) &lt; K_3$</td>
<td>$S(T) - K_1$</td>
<td>0</td>
<td>$-2(S(T) - K_2)$</td>
<td>$K_3 - S(T)$</td>
</tr>
<tr>
<td>$K_3 \leq S(T)$</td>
<td>$S(T) - K_1$</td>
<td>$S(T) - K_3$</td>
<td>$-2(S(T) - K_2)$</td>
<td>0</td>
</tr>
</tbody>
</table>
Butterfly Spread with Calls

Payoff/Profit

\[ C_1 - 2C_2 + C_3 \]

\[ K_1, K_2, K_3 \]

\[ S(T) \]
Example: Butterfly Spread

Example
Suppose we create a butterfly spread purchasing a call options with strike prices $150 and $200 selling 2 call options with strike price $175. Suppose that $C(150) = 60$, $C(175) = 35$, and $C(200) = 10$. Find the payoff and net profit if at expiry,

- $S = 100$
- $S = 150$
- $S = 175$
- $S = 200$
- $S = 250$. 
Straddles

Definition
Simultaneous long position in a call and a put is called a **long straddle**. Simultaneous short position in a call and a put is called a **short straddle**.

Remarks: We will assume
- the underlying stock is the same for both options,
- the strike prices are the same,
- the expiry dates are the same.
Suppose an investor has a long straddle with strike prices $K$.

<table>
<thead>
<tr>
<th>$S(T)$</th>
<th>Put Payoff</th>
<th>Call Payoff</th>
<th>Total Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(T) \leq K$</td>
<td>$K - S(T)$</td>
<td>0</td>
<td>$K - S(T)$</td>
</tr>
<tr>
<td>$K &lt; S(T)$</td>
<td>0</td>
<td>$S(T) - K$</td>
<td>$S(T) - K$</td>
</tr>
</tbody>
</table>
Long Straddle (2 of 2)

Payoff/Profit

\[ \text{Payoff} = (K-H) + \text{Profit} \]

\[ \text{Profit} = \begin{cases} +P & \text{if } S(T) > K+H \text{ or } S(T) < K-H \\ -P & \text{otherwise} \end{cases} \]
Example: Long Straddle

Example
Suppose we create a long straddle purchasing call and put options with strike prices of $150. Suppose that $C(150) = 20$ and $P(150) = 10$. Find the payoff and net profit if at expiry,

- $S = 100$
- $S = 140$
- $S = 150$
- $S = 160$
- $S = 200$. 
Definition
A **long strangle** is a simultaneous long position in a call and a put. A **short strangle** is a simultaneous short position in a call and a put.

**Remarks:** We will assume
- the underlying stock is the same for both options,
- the expiry dates are the same,
- the strike prices may be different (this characteristic distinguishes a strangle from a straddle).
Suppose an investor has a long strangle with strike prices $K_P$ for the put and $K_C > K_P$ for the call.

<table>
<thead>
<tr>
<th>$S(T)$</th>
<th>Put Payoff</th>
<th>Call Payoff</th>
<th>Total Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(T) \leq K_P$</td>
<td>$K_P - S(T)$</td>
<td>0</td>
<td>$K_P - S(T)$</td>
</tr>
<tr>
<td>$K_P &lt; S(T) &lt; K_C$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$K_C &lt; S(T)$</td>
<td>0</td>
<td>$S(T) - K_C$</td>
<td>$S(T) - K_C$</td>
</tr>
</tbody>
</table>
Example: Long Strangle

Example
Suppose we create a long strangle purchasing a call option with a strike price of $150 and a put option with a strike price of $130. Suppose that $C(150) = 20$ and $P(130) = 10$. Find the payoff and net profit if at expiry,

- $S = 100$
- $S = 140$
- $S = 150$
- $S = 160$
- $S = 200$. 
These slides are adapted from the textbook,


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