Introduction to Options

MATH 472 *Financial Mathematics*

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Objectives

In this lesson we will learn:

- the common types of options,
- features of options,
- the arbitrage-free relationships between options.

Definitions and Terminology

Definition

An **option** is the right, but *not* the obligation, to buy or sell a security such as a stock for an agreed upon price at some time in the future.

strike price: agreed upon price for buying or selling.

expiry: deadline by which the option must be exercised (also known as **exercise time**, **strike time**, and **expiry date**).

call option: an option to buy a security (sometimes just called a **call**).

put option: an option to sell a security (a **put** for short).

Option Listing

AES Corporation: \$10.65 (03/20/2018, 14:35ET)

Expiry	Strike	Bid	Ask	Vol.	Last	Net Chng	Open Int.
08/17/2018	10c	0.95	1.05	0	1.07		478
08/17/2018	11c	0.40	0.50	1	0.50	−0.05	369
08/17/2018	10p	0.40	0.50	0	0.40		605
08/17/2018	11p	0.90	1.00	0	0.95		280

Some Types of Options

European option: can only be exercised at expiry.

American option: can be exercised at or before expiry.

Other types exist such as **Asian**, **Bermudan**, **look-back**, *etc.*

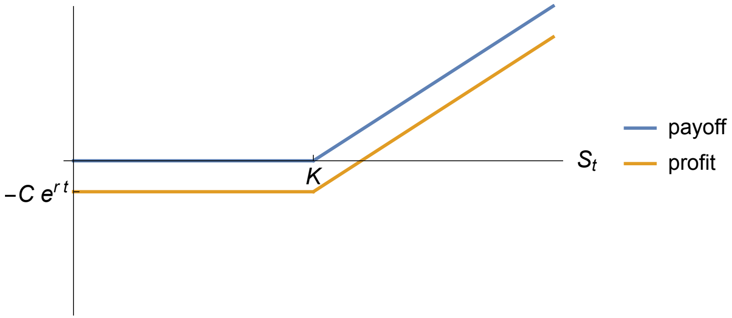
- Our goal over the next few class meetings is to determine a method for pricing the European-style options.
- Their values satisfy the **Black-Scholes partial differential equation.**

Notation

C^a: value of an American-style call option
C^e: value of a European-style call option
K: strike price of an option
P^a: value of an American-style put option
P^e: value of a European-style put option
r: continuously compounded, risk-free interest rate
δ: continuously compounded, dividend yield rate
S: price of a share of a security
T: exercise time or expiry of an option (sometimes called the strike time)
t: current time, generally with $0 \leq t \leq T$

Payoff/Profit for a Call Option

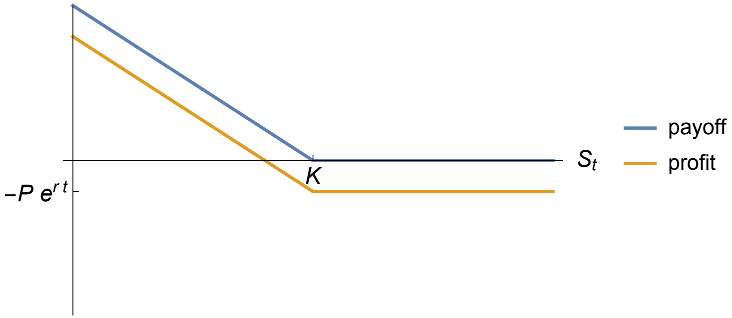
The **payoff** of a long K-strike call option on security S_t is $\max\{0, S_t - K\} = (S_t - K)^+$. A call option does not exhibit a positive payoff until the $S_t > K$.



The payoff of a call option minus the future value of its cost is the call's **profit**.

Payoff/Profit for a Put Option

The payoff for a long K-strike put option on security S_t is $\max\{0, K - S_t\} = (K - S_t)^+$. A put option does not exhibit a positive payoff until $S_t < K$.



The payoff of a put option minus the future value of its cost is the put's profit.

Properties of Options (1 of 3)

Theorem

$C^a \geq C^e$ and $P^a \geq P^e$.

An American form of an option will always be worth as much as the European version of the option (all other features being the same).

Proof

Assume: $C^a < C^e$.

- Sell the European option for C^e and purchase the American option for C^a.
- This generates cash flow $C^e - C^a > 0$ at time $t = 0$ which is invested at the risk-free rate r.
- If the European option holder chooses to exercise the option at expiry, the American option holder can also exercise.
- If the European option holder does not exercise, the American option holder can let the American option expire unused.
- At expiry the seller still holds $(C^e - C^a)e^{rT} > 0$.

Properties of Options (2 of 3)

Theorem

$C^e \geq S - K\,e^{-rT}$.

Remark: this is equivalent to the inequality

$$C^e + K\,e^{-rT} \geq S.$$

It can be interpreted as stating that the cost of a European call plus the present value of the strike is always at least the cost of the security.

Proof

Assume: $C^e < S - K\,e^{-rT}$.

- Short the security for S and purchase the European call for C^e.
- This generates cash flow $S - C^e$ at time $t = 0$. This is invested at the continuously compounded risk-free rate r.
- At expiry the option holder has a risk-free investment worth $(S - C^e)e^{rT}$ and spends no more than K to close out the short position in the security.
- At expiry the option holder has $(S - C^e)e^{rT} - K > 0$ since this inequality is equivalent to the assumption that $C^e < S - K\,e^{-rT}$.
- Therefore a risk-free positive profit can be had.

Example

What is the minimum price for a three-month call option on a non-dividend-paying stock when the stock price is \$30, the strike price is \$28, and the risk-free interest rate is 8%?

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$$C^e \geq S - K\,e^{-rT} = 30 - 28e^{-0.08(3/12)} = 2.5544$$

Properties of Options (3 of 3)

Theorem

$K\,e^{-rT} \leq P^e + S.$

Remark: this inequality can be interpreted as stating that the present value of the strike is at most the sum of the costs of the security and a put.

Proof

Assume: $K\,e^{-rT} > P^e + S$.

- Borrow $P^e + S$ to purchase the security for S and a European put for P^e. The net cash flow at $t = 0$ is zero.
- At expiry the investor will sell the security for at least K and must repay the loan in the amount $(P^e + S)e^{rT}$.
- At $t = T$ the net cash flow is at least $K - (P^e + S)e^{rT} > 0$.
- Therefore a risk-free positive profit can be had.

Example

What is the minimum price for a three-month European put option on a non-dividend-paying stock when the stock price is \$13, the strike price is \$15, and the risk-free interest rate is 7%?

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$$P^e \geq K\, e^{-rT} - S = 15e^{-0.07(3/12)} - 13 = 1.7398$$

Put-Call Parity Formula

Theorem

For non-dividend paying stocks, if the European put and call have the same strike price and expiry, then

$$P^e + S = C^e + K\, e^{-rT}.$$

Remark: Put-Call Parity can be interpreted as stating that the cost of a European put plus the security equals the cost of the European call plus the present value of the strike.

Proof (1 of 2)

Assume: $P^e + S < C^e + K\, e^{-rT}$.

- Borrow $P^e + S - C^e$ at the continuously compounded risk-free rate r to purchase the European put option for P^e, the security for S, and sell the European call option for C^e. The net cashflow at $t = 0$ is zero.
- At expiry sell the security for at least K and pay back the loan with interest.
- At expiry this leaves the borrower with $K - (P^e + S - C^e)e^{rT} > 0$ since this inequality is equivalent to the assumption that $P^e + S < C^e + Ke^{-rT}$.
- Therefore a risk-free positive profit can be had.

Proof (2 of 2)

Assume: $P^e + S > C^e + K\,e^{-rT}$.

- Short the security for S, sell the European put option for P^e, and purchase the European call option for C^e. At time $t = 0$ this generates a cash flow of $S + P^e - C^e > 0$. Invest this amount at the continuously compounded risk-free rate r.
- At expiry purchase the security for at most K and close out the short position.
- At expiry the short seller holds $(S + P^e - C^e)e^{rT} - K > 0$ since this inequality is equivalent to the assumption that $P^e + S > C^e + K\,e^{-rT}$.

Example

The price of a \$30-strike, six-month European call is \$2. The stock price is \$29 and the risk-free interest rate is 10%. What is the price of \$30-strike, six-month European put?

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$$P^e = C^e + K\, e^{-rT} - S = 2 + 30e^{-0.10(6/12)} - 29 = 1.5369$$

Effect of Dividends

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- The amount of the dividend will be $\delta S(t_d)$.

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- Consider the one-sided limits:

$$\begin{aligned}\lim_{t \to t_d^-} S(t) &= S(t_d^-) \\ \lim_{t \to t_d^+} S(t) &= S(t_d^+).\end{aligned}$$

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- In the absence of arbitrage,

$$S(t_d^+) = (1 - \delta) S(t_d^-).$$

Discrete Dividends

Theorem

If n dividend payments of the form $\delta S(t_i^-)$ will be made at times t_i^- for $i = 1, 2, \ldots, n$ then the Put-Call Parity Formula for discrete dividend payments can be expressed as

$$P^e + S(0) - \delta \sum_{i=1}^{n} S(t_i^-) e^{-rt_i} = C^e + K\, e^{-rT}.$$

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$$P^e + S(0) - \delta \sum_{i=1}^{n} S(t_i^-) e^{-rt_i} = C^e + K\, e^{-rT}.$$

The value of the security is discounted by the total of the present values of the dividends paid.

Example

What is the minimum price for a \$60-strike, four-month European call option on a stock selling for \$64. The stock will pay a dividend of \$1 per share in one month. The risk-free interest rate is 6%.

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\begin{aligned}
P^e + S(0) - D\,e^{-rt_i} &= C^e + K\,e^{-rT} \\
P^e + S(0) - D\,e^{-rt_i} - K\,e^{-rT} &= C^e \\
S(0) - D\,e^{-rt_i} - K\,e^{-rT} &\leq C^e \\
64 - (1)e^{-0.06(1/12)} - 60e^{-0.06(4/12)} = 4.1931 &\leq C^e
\end{aligned}
$$

Continuous Dividends

Theorem

For European options on securities which pay dividends at a continuous, constant dividend yield δ*, the Put-Call Parity Formula takes on the form*

$$P^e + S(0)e^{-\delta T} = C^e + K\,e^{-rT}.$$

Homework

- Read Sections 7.1, 7.2
- Exercises: 1–9

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

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