Options on Continuous Dividend Securities

MATH 472 *Financial Mathematics*

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Objectives

In this lesson we will learn to:

- model the value of a security paying dividends at a continuous rate,
- price European options on securities that pay dividends at a continuous rate.

Dividends

- We have versions of the Put-Call Parity formula which include the effects of dividends:

$$P^e + Se^{-\delta T} = C^e + K\, e^{-r\, T} \quad \text{(continuous)}$$

$$P^e + S(0) - \delta \sum_{i=1}^{n} S(t_i^-) e^{-r\, t_i} = C^e + K\, e^{-r\, T} \quad \text{(discrete)}$$

- We do not have pricing formulas for the options themselves. We explore modifications and extensions to the Black-Scholes partial differential equation and its solution in this lesson.

Basic Problem for the European Call

The non-dividend-paying stock is assumed to obey the stochastic process

$$dS = \mu\, S\, dt + \sigma\, S\, dW(t)$$

and the European call solves the initial boundary value problem:

$$\begin{aligned} r\,F &= F_t + r\,S\,F_S + \frac{1}{2}\sigma^2 S^2 F_{SS} \quad \text{for } (S,t) \text{ in } [0,\infty)\times[0,T], \\ F(S,T) &= (S(T)-K)^+ \quad \text{for } S > 0, \\ F(0,t) &= 0 \quad \text{for } 0 \le t < T, \\ F(S,t) &= S - K\,e^{-r(T-t)} \quad \text{as } S \to \infty. \end{aligned}$$

Stock Pays Continuous Dividends

Assumption: the stock pays dividends at a continuous rate proportional to the value of the stock

- What is a suitable expression for the dividend yield (dividend paid per unit time)?

- How much dividend is paid in a short time interval dt?

- What stochastic differential equation would the value of the stock paying a continuous proportional dividend obey?

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$$\text{dividend per unit time} = \delta\, S$$

- How much dividend is paid in a short time interval dt?

$$\text{dividend paid} = \delta\, S\, dt$$

- What stochastic differential equation would the value of the stock paying a continuous proportional dividend obey?

$$dS = (\mu - \delta) S\, dt + \sigma\, S\, dW(t)$$

Suppose $F(S,t)$ is the value of a European call option on the stock paying a continuous dividend. F obeys the following stochastic differential equation:

$$dF = \left((\mu - \delta) S\, F_S + \frac{1}{2}\sigma^2 S^2 F_{SS} + F_t\right) dt + \sigma\, S\, F_S\, dW(t).$$

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As before, we wish to eliminate the random part of this equation by creating a portfolio of a long position in the call option and a short position in Δ shares of the stock.

$$\Pi = F - (\Delta) S$$

Change in Portfolio Value

One share of stock pays $\delta S\,dt$ in dividends during a time interval of length dt, thus Δ shares of stock pays

$$\delta(\Delta) S\,dt \quad \text{in dividends.}$$

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The portfolio changes in value

$$\begin{aligned}
d\Pi &= d(F - (\Delta)S) - \delta(\Delta)S\,dt \\
&= dF - (\Delta)dS - \delta(\Delta)S\,dt \\
&= \left((\mu - \delta)S\,F_S + \frac{1}{2}\sigma^2 S^2 F_{SS} + F_t\right) dt + \sigma\,S\,F_S\,dW(t) \\
&\quad - (\Delta)\left((\mu - \delta)S\,dt + \sigma\,S\,dW(t)\right) - \delta(\Delta)S\,dt \\
&= \left((\mu - \delta)S(F_S - \Delta) + \frac{1}{2}\sigma^2 S^2 F_{SS} + F_t - \delta(\Delta)S\right) dt \\
&\quad + \sigma S(F_S - \Delta)\,dW(t).
\end{aligned}$$

Eliminating Randomness

Choose $\Delta = F_S$ and the portfolio obeys the stochastic differential equation:

$$d\Pi = \left(\frac{1}{2}\sigma^2 S^2 F_{SS} + F_t - \delta\, S\, F_S\right)\, dt.$$

In the absence of arbitrage the change in the value of the portfolio should be the same as the interest earned by a equivalent amount of cash.

$$d\Pi = r(F - (\Delta)S)\, dt$$

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$$d\Pi = r(F - (\Delta)S)\, dt$$

Thus the Black-Scholes partial differential equation for the stock paying continuous dividends becomes

$$r\, F = F_t + \frac{1}{2}\sigma^2 S^2 F_{SS} + (r - \delta) S\, F_S.$$

Similarities with Non-Dividend-Paying Stocks

- Payoff of the call option at expiry: $F(S, T) = (S(T) - K)^+$.
- Boundary condition at $S = 0$ is $F(0, t) = 0$.
- Boundary condition as $S \to \infty$:

$$\begin{aligned}
F(S,t) &= P^e + S\,e^{-\delta(T-t)} - K\,e^{-r(T-t)} \\
\lim_{S\to\infty} F(S,t) &= \lim_{S\to\infty} \left(P^e + S\,e^{-\delta(T-t)} - K\,e^{-r(T-t)}\right) \\
&= S\,e^{-\delta(T-t)} - K\,e^{-r(T-t)}.
\end{aligned}$$

Change of Variables

Define the function $G(S,t) = e^{\delta(T-t)} F(S,t)$, then

$$\begin{aligned}
G(S,T) &= e^{\delta(T-T)} F(S,T) = (S(T) - K)^+ \\
G(0,t) &= e^{\delta(T-t)} F(0,t) = 0 \\
\lim_{S\to\infty} G(S,t) &= e^{\delta(T-t)} \left(S\, e^{-\delta(T-t)} - K\, e^{-r(T-t)} \right) \\
&= S - K\, e^{-(r-\delta)(T-t)} \\
F_S &= e^{-\delta(T-t)} G_S \\
F_{SS} &= e^{-\delta(T-t)} G_{SS} \\
F_t &= e^{-\delta(T-t)} (\delta\, G + G_t).
\end{aligned}$$

Substitute these expressions into the partial differential equation, boundary conditions, and the final condition for the European call option on the stock paying continuous dividends.

Initial Boundary Value Problem

$$\begin{aligned}
(r-\delta)G &= G_t + \frac{1}{2}\sigma^2 S^2 G_{SS} + (r-\delta)S\,G_S \\
G(S,T) &= (S(T)-K)^+ \\
G(0,t) &= 0 \\
\lim_{S\to\infty} G(S,t) &= S - K\,e^{-(r-\delta)(T-t)}
\end{aligned}$$

Remark: this is exactly the same initial boundary value problem we have already solved except r has been replaced by $r-\delta$.

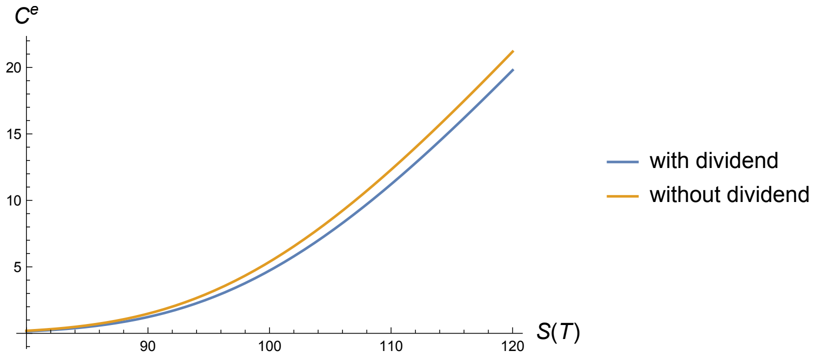
Black-Scholes Option Pricing Formulas

For a stock paying a continuous, proportional dividend at rate δ the value of a European options are given by the formulas

$$
\begin{aligned}
w &= \frac{\ln(S/K) + (r - \delta + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \\
C^{e,\delta} &= S\, e^{-\delta(T-t)}\Phi\,(w) - K\, e^{-r(T-t)}\Phi\left(w - \sigma\sqrt{T - t}\right) \\
P^{e,\delta} &= K\, e^{-r(T-t)}\Phi\left(-w + \sigma\sqrt{T - t}\right) - S\, e^{-\delta(T-t)}\Phi\,(-w)
\end{aligned}
$$

Comparison

$S(0) = 100$, $\delta = 0.05$, $r = 0.0325$, $\sigma = 0.25$, $K = 100$, $T = 3/12$, $t = 0$.



Example: Call Option

Suppose the current price of a security is \$62 per share. The continuously compounded interest rate is 10% per year. The volatility of the price of the security is $\sigma = 20\%$ per year. The stock pays dividends continuously at a rate of $\delta = 3\%$ per year. Find the cost of a five-month European call option with a strike price of \$60 per share.

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$$T = \frac{5}{12}, \quad t = 0, \qquad r = 0.10, \quad \sigma = 0.20,$$
$$S = 62, \quad K = 60, \qquad \delta = 0.03$$

Using the formula for w and $C^{e,\delta}$ we have

$$\begin{aligned} w &\approx 0.544463 \\ C^{e,\delta} &\approx \$5.24 \end{aligned}$$

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Without the dividend we calculate $C^e \approx \$5.80$.

Example: Put Option

Suppose the current price of a security is \$97 per share. The stock pays a continuous dividend at a yield of 6.5% per year. The continuously compounded interest rate is 8% per year. The volatility of the price of the security is $\sigma = 45\%$ per year. Find the cost of a three-month European put option with a strike price of \$95 per share.

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Suppose the current price of a security is \$97 per share. The stock pays a continuous dividend at a yield of 6.5% per year. The continuously compounded interest rate is 8% per year. The volatility of the price of the security is $\sigma = 45\%$ per year. Find the cost of a three-month European put option with a strike price of \$95 per share.

$$
\begin{aligned}
T &= 1/4, \quad t = 0, \quad r = 0.08, \quad \sigma = 0.45, \\
\delta &= 0.065, \quad S = 97, \quad K = 95.
\end{aligned}
$$

Using the formulas for w and $P^{e,\delta}$ we obtain

$$
\begin{aligned}
w &\approx 0.221763 \\
P^{e,\delta} &\approx \$7.34
\end{aligned}
$$

Useful Result

Lemma

$$S\,e^{-\delta(T-t)}\phi\,(w) = K\,e^{-r(T-t)}\phi\left(w - \sigma\sqrt{T-t}\right)$$

A New Greek

The rate of change in the price of a European call option on a stock paying continuous dividends is

$$\Psi_{C^{e,\delta}} = \frac{\partial C^{e,\delta}}{\partial \delta} = -S(T-t)e^{-\delta(T-t)}\Phi(w) < 0.$$

For a European put option

$$\Psi_{P^{e,\delta}} = \frac{\partial P^{e,\delta}}{\partial \delta} = S(T-t)e^{-\delta(T-t)}(1-\Phi(w)) > 0.$$

Example

The current price of a security is \$50, the risk-free interest rate is 6%, the volatility of the security is 40%, the strike price of a European call option on the security will be \$52, the continuous dividend yield of the security is 1%, and the strike time of the option is 3 months.

Estimate the change in the price of the European call option if the continuous dividend rate is lowered to 0.90%.

Solution

$$
\begin{aligned}
w &= \frac{\ln\frac{50}{52} + \left(0.06 - 0.01 + \frac{(0.40)^2}{2}\right)(3/12 - 0)}{0.40\sqrt{3/12 - 0}} \approx -0.0336036 \\
\Psi_{C^{e,\delta}} &= -50(3/12 - 0)e^{-0.01(3/12-0)}\Phi(w) \approx -6.06727 \\
dC^{e,\delta} &\approx (-6.06727)(-0.001) = \$0.00606727
\end{aligned}
$$

Dividends Influence on Other Greeks

The presence of the continuous dividend rate δ, in the Call and Put formulas alters the previously discussed Greeks.

$$
\begin{aligned}
w &= \frac{\ln(S/K) + (r - \delta + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \\
C^{e,\delta} &= S\, e^{-\delta(T-t)} \Phi\,(w) - K\, e^{-r(T-t)} \Phi\left(w - \sigma\sqrt{T - t}\right) \\
P^{e,\delta} &= K\, e^{-r(T-t)} \Phi\left(-w + \sigma\sqrt{T - t}\right) - S\, e^{-\delta(T-t)} \Phi\,(-w)
\end{aligned}
$$

Find Delta, Gamma, Rho, Theta, and Vega.

Delta

For a Call:

$$\begin{aligned}\frac{\partial C^{e,\delta}}{\partial S} &= e^{-\delta(T-t)}\Phi\left(w\right) + S\,e^{-\delta(T-t)}\phi\left(w\right)\frac{\partial w}{\partial S} \\ &\quad - K\,e^{-r(T-t)}\phi\left(w-\sigma\sqrt{T-t}\right)\frac{\partial w}{\partial S} \\ &= e^{-\delta(T-t)}\Phi\left(w\right) > 0.\end{aligned}$$

Delta

For a Call:

$$\begin{aligned}\frac{\partial C^{e,\delta}}{\partial S} &= e^{-\delta(T-t)}\Phi(w) + S\,e^{-\delta(T-t)}\phi(w)\frac{\partial w}{\partial S} \\ &\quad - K\,e^{-r(T-t)}\phi\left(w-\sigma\sqrt{T-t}\right)\frac{\partial w}{\partial S} \\ &= e^{-\delta(T-t)}\Phi(w) > 0.\end{aligned}$$

For a Put:

$$\begin{aligned}\frac{\partial P^{e,\delta}}{\partial S} &= -K\,e^{-r(T-t)}\phi\left(-w+\sigma\sqrt{T-t}\right)\frac{\partial w}{\partial S} - e^{-\delta(T-t)}\Phi(-w) \\ &\quad + S\,e^{-\delta(T-t)}\phi(-w)\frac{\partial w}{\partial S} \\ &= -e^{-\delta(T-t)}\Phi(-w) \\ &= e^{-\delta(T-t)}\left(\Phi(w)-1\right) < 0.\end{aligned}$$

Gamma

For a Call:

$$\begin{aligned}
\frac{\partial^2 C^{e,\delta}}{\partial S^2} &= e^{-\delta(T-t)} \phi(w) \frac{\partial w}{\partial S} \\
&= e^{-\delta(T-t)} \frac{\phi(w)}{\sigma S \sqrt{T-t}} \\
&= e^{-\delta(T-t)} \frac{e^{-w^2/2}}{\sigma S \sqrt{2\pi(T-t)}} > 0.
\end{aligned}$$

Gamma

For a Call:

$$\begin{aligned}\frac{\partial^2 C^{e,\delta}}{\partial S^2} &= e^{-\delta(T-t)}\phi(w)\frac{\partial w}{\partial S} \\ &= e^{-\delta(T-t)}\frac{\phi(w)}{\sigma S\sqrt{T-t}} \\ &= e^{-\delta(T-t)}\frac{e^{-w^2/2}}{\sigma S\sqrt{2\pi(T-t)}} > 0.\end{aligned}$$

For a Put:

$$\begin{aligned}\frac{\partial^2 P^{e,\delta}}{\partial S^2} &= e^{-\delta(T-t)}\phi(w)\frac{\partial w}{\partial S} \\ &= e^{-\delta(T-t)}\frac{\phi(w)}{\sigma S\sqrt{T-t}} \\ &= e^{-\delta(T-t)}\frac{e^{-w^2/2}}{\sigma S\sqrt{2\pi(T-t)}} > 0.\end{aligned}$$

Rho

For a Call:

$$
\begin{aligned}
\frac{\partial C^{e,\delta}}{\partial r} &= S\,e^{-\delta(T-t)}\phi\,(w)\frac{\partial w}{\partial r} + K(T-t)e^{-r(T-t)}\Phi\left(w-\sigma\sqrt{T-t}\right) \\
&\quad - Ke^{-r(T-t)}\phi\left(w-\sigma\sqrt{T-t}\right)\frac{\partial w}{\partial r} \\
&= K(T-t)e^{-r(T-t)}\Phi\left(w-\sigma\sqrt{T-t}\right) > 0.
\end{aligned}
$$

Rho

For a Call:

$$\begin{aligned}\frac{\partial C^{e,\delta}}{\partial r} &= S\,e^{-\delta(T-t)}\phi\,(w)\frac{\partial w}{\partial r} + K(T-t)e^{-r(T-t)}\Phi\left(w-\sigma\sqrt{T-t}\right)\\ &\quad - Ke^{-r(T-t)}\phi\left(w-\sigma\sqrt{T-t}\right)\frac{\partial w}{\partial r}\\ &= K(T-t)e^{-r(T-t)}\Phi\left(w-\sigma\sqrt{T-t}\right) > 0.\end{aligned}$$

For a Put:

$$\begin{aligned}\frac{\partial P^{e,\delta}}{\partial r} &= -K(T-t)e^{-r(T-t)}\Phi\left(-w+\sigma\sqrt{T-t}\right)\\ &\quad - K\,e^{-r(T-t)}\phi\left(-w+\sigma\sqrt{T-t}\right)\frac{\partial w}{\partial r} + S\,e^{-\delta(T-t)}\phi\,(-w)\frac{\partial w}{\partial r}\\ &= -K(T-t)e^{-r(T-t)}\Phi\left(-w+\sigma\sqrt{T-t}\right) < 0.\end{aligned}$$

Theta (1 of 2)

For a Call:

$$
\begin{aligned}
\frac{\partial C^{e,\delta}}{\partial t} &= \delta\, S\, e^{-\delta(T-t)} \Phi\,(w) + S\, e^{-\delta(T-t)} \phi\,(w) \frac{\partial w}{\partial t} \\
&\quad - K\, r\, e^{-r(T-t)} \Phi\left(w - \sigma\sqrt{T-t}\right) \\
&\quad - K\, e^{-r(T-t)} \phi\left(w - \sigma\sqrt{T-t}\right) \left(\frac{\partial w}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}\right) \\
&= \delta\, S\, e^{-\delta(T-t)} \Phi\,(w) - K\, r\, e^{-r(T-t)} \Phi\left(w - \sigma\sqrt{T-t}\right) \\
&\quad - S\, e^{-\delta(T-t)} \frac{\sigma e^{-w^2/2}}{2\sqrt{2\pi(T-t)}}.
\end{aligned}
$$

Theta (2 of 2)

For a Put:

$$\begin{aligned}
\frac{\partial P^{e,\delta}}{\partial t} &= K\, r\, e^{-r(T-t)} \Phi\left(-w + \sigma\sqrt{T-t}\right) \\
&\quad - K\, e^{-r(T-t)} \phi\left(-w + \sigma\sqrt{T-t}\right) \left(\frac{\partial w}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}\right) \\
&\quad - \delta\, S\, e^{-\delta(T-t)} \Phi(-w) + S\, e^{-\delta(T-t)} \phi(-w) \frac{\partial w}{\partial t} \\
&= K\, r\, e^{-r(T-t)} \Phi\left(-w + \sigma\sqrt{T-t}\right) - \delta\, S\, e^{-\delta(T-t)} \Phi(-w) \\
&\quad - S\, e^{-\delta(T-t)} \frac{\sigma e^{-w^2/2}}{2\sqrt{2\pi(T-t)}}.
\end{aligned}$$

Vega

For a Call:

$$\begin{aligned}\frac{\partial C^{e,\delta}}{\partial \sigma} &= S\,e^{-\delta(T-t)}\phi\,(w)\frac{\partial w}{\partial \sigma} \\ &\quad - K\,e^{-r(T-t)}\phi\left(w-\sigma\sqrt{T-t}\right)\left(\frac{\partial w}{\partial \sigma}-\sqrt{T-t}\right) \\ &= e^{-\delta(T-t)}\frac{S\sqrt{T-t}}{\sqrt{2\pi}}e^{-w^2/2} > 0.\end{aligned}$$

Vega

For a Call:

$$\begin{aligned}\frac{\partial C^{e,\delta}}{\partial \sigma} &= S\,e^{-\delta(T-t)}\phi(w)\frac{\partial w}{\partial \sigma} \\ &\quad - K\,e^{-r(T-t)}\phi\left(w-\sigma\sqrt{T-t}\right)\left(\frac{\partial w}{\partial \sigma}-\sqrt{T-t}\right) \\ &= e^{-\delta(T-t)}\frac{S\sqrt{T-t}}{\sqrt{2\pi}}e^{-w^2/2} > 0.\end{aligned}$$

For a Put:

$$\begin{aligned}\frac{\partial P^{e,\delta}}{\partial \sigma} &= -K\,e^{-r(T-t)}\phi\left(-w+\sigma\sqrt{T-t}\right)\left(\frac{\partial w}{\partial \sigma}-\sqrt{T-t}\right) \\ &\quad + S\,e^{-\delta(T-t)}\phi(-w)\frac{\partial w}{\partial \sigma} \\ &= e^{-\delta(T-t)}\frac{S\sqrt{T-t}}{\sqrt{2\pi}}e^{-w^2/2} > 0.\end{aligned}$$

Homework

- Read Section 11.1
- Exercises: 1–8

Credits

These slides are adapted from the textbook,

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