The Spot Rate

MATH 472 *Financial Mathematics*

J Robert Buchanan

2018

Objectives

In this lesson we will learn:

- to calculate present and future value in the context of time-varying interest rates,
- how to find the yield curve,
- how to find the present and future values of continuous income streams,
- how to calculate the internal rate of return of an investment.

Continuously Varying Interest Rates (1 of 2)

Definition

If interest is compounded continuously at a time-dependent rate $r(t)$, the function $r(t)$ is referred to as the **spot rate.**

Continuously Varying Interest Rates (1 of 2)

Definition

If interest is compounded continuously at a time-dependent rate $r(t)$, the function $r(t)$ is referred to as the **spot rate.**

- Suppose the amount due at $t = 0$ is $A(0) = 1$.

Continuously Varying Interest Rates (1 of 2)

Definition

If interest is compounded continuously at a time-dependent rate $r(t)$, the function $r(t)$ is referred to as the **spot rate.**

- Suppose the amount due at $t = 0$ is $A(0) = 1$.
- The amount due at time t is $A(t)$ and if Δt is small then

$$\begin{aligned} A(t + \Delta t) &\approx A(t)(1 + r(t)\Delta t) \\ \frac{A(t + \Delta t) - A(t)}{\Delta t} &\approx r(t)A(t) \\ A'(t) &= r(t)A(t). \end{aligned}$$

Continuously Varying Interest Rates (1 of 2)

Definition

If interest is compounded continuously at a time-dependent rate $r(t)$, the function $r(t)$ is referred to as the **spot rate.**

- Suppose the amount due at $t = 0$ is $A(0) = 1$.
- The amount due at time t is $A(t)$ and if Δt is small then

$$\begin{aligned} A(t + \Delta t) &\approx A(t)(1 + r(t)\Delta t) \\ \frac{A(t + \Delta t) - A(t)}{\Delta t} &\approx r(t)A(t) \\ A'(t) &= r(t)A(t). \end{aligned}$$

- The function $r(t) = \dfrac{A'(t)}{A(t)}$ is also known as the **force of interest.**

Continuously Varying Interest Rates (2 of 2)

Amount due at time $t > 0$ on a unit (\$1) deposit:

$$a(t) = e^{\int_0^t r(s)\,ds}.$$

Time t future value of $A(0)$:

$$A(t) = A(0)e^{\int_0^t r(s)\,ds} = A(0)a(t).$$

Continuously Varying Interest Rates (2 of 2)

Amount due at time $t > 0$ on a unit ($1) deposit:

$$a(t) = e^{\int_0^t r(s)\,ds}.$$

Time t future value of $A(0)$:

$$A(t) = A(0)e^{\int_0^t r(s)\,ds} = A(0)a(t).$$

Present value of a unit ($1) due at time $t > 0$:

$$v(t) = e^{-\int_0^t r(s)\,ds}$$

Present value of F:

$$F_0 = F\,e^{-\int_0^t r(s)\,ds} = F\,v(t).$$

Yield Curve

Definition

The average of the spot rate over the interval $[0, t]$

$$\bar{r}(t) = \frac{1}{t}\int_0^t r(s)\,ds$$

is called the **yield curve.**

Note that $\bar{r}(t)t = \int_0^t r(s)\,ds$ and thus

$$\begin{aligned} a(t) &= e^{\bar{r}(t)t} \\ v(t) &= e^{-\bar{r}(t)t}. \end{aligned}$$

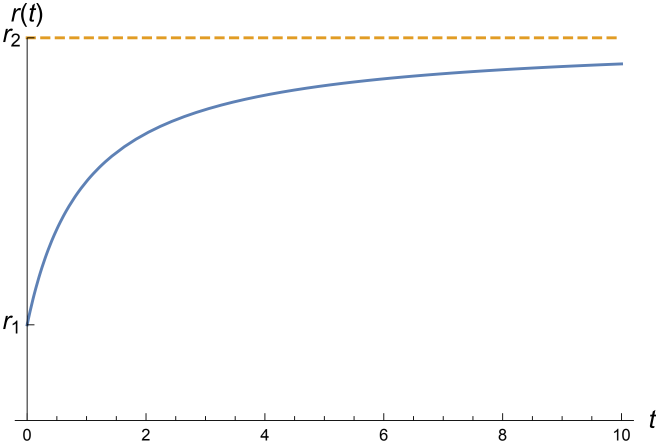
Example

Suppose the spot rate is $r(t) = \frac{r_1}{1+t} + \frac{r_2 t}{1+t}$.

1. Find the yield curve $\overline{r}(t)$.
2. Find the future value of a unit deposit at time $t > 0$.
3. Find the present value of a unit due at time $t > 0$.

Illustration of Spot Rate

If $0 < r_1 < r_2$ then the spot rate resembles the following.



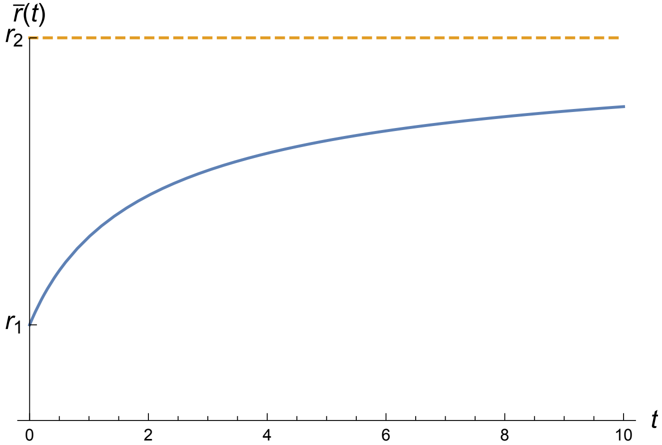
Solution (1 of 3)

Yield curve:

$$
\begin{aligned}
\bar{r}(t) &= \frac{1}{t}\int_0^t \left(\frac{r_1}{1+s} + \frac{r_2 s}{1+s} \right) ds \\
&= \frac{r_1}{t} \ln(1+t) + \frac{r_2}{t} \left(t - \ln(1+t) \right) \\
&= r_2 + \frac{r_1 - r_2}{t} \ln(1+t)
\end{aligned}
$$

Illustration of Yield Curve

If $0 < r_1 < r_2$ then the yield curve resembles the following.



Example (2 of 3)

Future value of a unit deposit:

$$\begin{aligned} a(t) &= e^{\int_0^t r(s)\,ds} \\ &= e^{t\bar{r}(t)} \\ &= e^{t\left(r_2+\frac{r_1-r_2}{t}\ln(1+t)\right)} \\ &= e^{r_2 t+(r_1-r_2)\ln(1+t)} \\ a(t) &= (1+t)^{r_1-r_2} e^{r_2 t} \end{aligned}$$

Example (3 of 3)

Present value of a unit amount:

$$\begin{aligned}
v(t) &= e^{-\int_0^t r(s)\,ds} \\
&= e^{-t\bar{r}(t)} \\
&= e^{-t\left(r_2+\frac{r_1-r_2}{t}\ln(1+t)\right)} \\
&= e^{-r_2 t-(r_1-r_2)\ln(1+t)} \\
v(t) &= (1+t)^{r_2-r_1} e^{-r_2 t}
\end{aligned}$$

Continuous Income Streams

Suppose the income received per unit time is the function $S(t)$ for $a \leq t \leq b$.

A **Riemann sum** approximates the total income received

$$\sum_{k=1}^{n} S(t_k)(t_k - t_{k-1}).$$

As $n \to \infty$ the total income is

$$S_{\text{tot}} = \int_a^b S(t)\,dt.$$

Amount Due and Present Value

If the continuously compounded interest rate is $r(t)$, the present value at time $t = 0$ of the income stream $S(t)$ for $0 \leq t \leq T$ is

$$P = \int_0^T e^{-r(t)\,t} S(t)\,dt.$$

The future value at $t = T$ of the income stream is

$$A = \int_0^T e^{r(t)(T-t)} S(t)\,dt.$$

Example

Suppose the slot machine floor of a new casino is expected to bring in $\$30,000$ per day. What is the present value of the first year's slot machine revenue assuming the continuously compounded annual interest rate is 3.55%?

Example

Suppose the slot machine floor of a new casino is expected to bring in $\$30,000$ per day. What is the present value of the first year's slot machine revenue assuming the continuously compounded annual interest rate is 3.55%?

Solution

Note that \$30,000/day is \$(30000)(365)/year.

$$\begin{aligned} P &= \int_0^1 (30000)(365)e^{-0.0355t}\,dt \\ &= \left[\frac{(30000)(365)}{-0.0355}e^{-0.0355t}\right]_{t=0}^{t=1} \approx \$10,757,917.19 \end{aligned}$$

Rate of Return

Definition

If an investment of amount P now receives an amount due of A one time unit from now, the **rate of return** (denoted r) is the equivalent interest rate so that the present value of A is P.

$$P = A(1 + r)^{-1}$$

Example

If you loan a friend \$100 today with the understanding that they will pay you back \$110 in one year's time, what is the rate of return?

Example

If you loan a friend \$100 today with the understanding that they will pay you back \$110 in one year's time, what is the rate of return?

Solution

$$
\begin{aligned}
P &= A(1+r)^{-1} \\
100 &= 110(1+r)^{-1} \\
1+r &= \frac{110}{100} \\
r &= 0.10
\end{aligned}
$$

General Setting

Suppose you invest an amount P now and receive a sequence of positive payoffs $\{A_1, A_2, \ldots, A_n\}$ at regular intervals. The rate of return per period is the interest rate such that the present value of the sequence of payoffs is equal to the amount invested.

$$P = \sum_{i=1}^{n} A_i (1 + r)^{-i}.$$

Example

Suppose you loan a friend \$100 with the agreement that they will pay you at the end of each year for the next five years amounts $\{21, 22, 23, 24, 25\}$. Find the annual rate of return.

Example

Suppose you loan a friend \$100 with the agreement that they will pay you at the end of each year for the next five years amounts $\{21, 22, 23, 24, 25\}$. Find the annual rate of return.

Solution

$$\begin{aligned} 100 &= \frac{21}{1+r} + \frac{22}{(1+r)^2} + \frac{23}{(1+r)^3} + \frac{24}{(1+r)^4} + \frac{25}{(1+r)^5} \\ r &\approx 0.0470299 \end{aligned}$$

The solution to the equation is approximated using Newton's method with an initial approximation of 0.03*.*

Example: Harvesting a Crop

Suppose you can stock a pond with fish that you can later sell for food. Stocking the pond requires an initial outlay of capital, but once stocked the fish and pond are self-sustaining. You can choose when the harvest the fish, but the longer you wait to harvest, the larger the fish will be. The annually compounded interest rate is 5%. If you harvest after one year the cash flow stream is $\{-100, 200\}$. If you harvest after two years the cash flow stream is $\{-100, 0, 250\}$. Using the rate of return as the basis for the decision, when should you harvest?

Solution

- Harvest after one year:

$$100 = \frac{200}{1+r} \iff r = 1 \text{ or } 100\%.$$

- Harvest after two years:

$$100 = \frac{250}{(1+r)^2} \iff r = 0.5811 \text{ or } 58.11\%.$$

Homework

- Read Sections 1.5–1.7.
- Exercises: 11–16.

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

author: J. Robert Buchanan

publisher: World Scientific Publishing Co. Pte. Ltd.

address: 27 Warren St., Suite 401–402, Hackensack, NJ 07601

ISBN: 978-9814407441