Stochastic Differential Equations

MATH 472 *Financial Mathematics*

J. Robert Buchanan

2018

Objectives

In this lesson we will learn:

- the definition of the stochastic integral,
- properties of the stochastic integral,
- differences between Riemann and stochastic integrals,
- how to solve elementary stochastic differential equations,
- Itô's Lemma for changing variables in stochastic differential equations.

Review of Wiener Process

Earlier we developed the continuous random process $\{W_t\}_{t \geq 0}$ as the limit if a discrete random walk.

- W_t is a continuous function of t,
- $W_0 = 0$ with probability one,
- W_t is normally distributed with $\mathbb{E}(W_t) = 0$ and $\mathrm{Var}(W_t) = \sigma^2 t$.
- If $t_1 < t_2 < t_3$ then $W_{t_2} - W_{t_1}$ and $W_{t_3} - W_{t_2}$ are independent.
- If $s < t$ then $\mathbb{E}(W_t - W_s) = 0$ and $\mathrm{Var}(W_t - W_s) = \sigma^2(t - s)$.

Differential Wiener Process

Let $\sigma^2 = 1$ and suppose we denote $\Delta W_t = W_{t+\Delta t} - W_t$ for $\Delta t > 0$.

$$\mathrm{Var}\,(\Delta W_t) = \mathbb{E}\left((\Delta W_t)^2\right) = \Delta t.$$

This relationship holds as $\Delta t \to 0^+$, thus we write

$$(dW_t)^2 = dt.$$

The expression dW_t is the **differential of** W_t.

Integral of a Wiener Process

The **stochastic integral** of $f(x)$ on the interval $[0, t]$ is defined to be

$$\int_0^t f(\tau)\, dW_\tau = \lim_{n \to \infty} \sum_{k=1}^{n} f(t_{k-1}) \left(W_{t_k} - W_{t_{k-1}} \right)$$

where $t_k = \dfrac{k\, t}{n}$.

Remarks

- $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k < \cdots < t_n = t$ is a partition of $[0, t]$.
- The function f is evaluated at the left-hand endpoint of each subinterval.
- Since $t_{k-1} < t_k$ the future value of W_{t_k} is still a random variable.

If $Z(t) = \int_0^t f(\tau)\, dW_\tau$ then $dZ = f(t)\, dW_t$.

Properties of Stochastic Integrals

Theorem

If f is a deterministic (non-random) function defined on $[0, t]$ for which

$$\int_0^t f(\tau)\, dW_\tau$$

exists, then

$$\begin{aligned} \mathbb{E}\left(\int_0^t f(\tau)\, dW_\tau\right) &= 0 \\ \mathrm{Var}\left(\int_0^t f(\tau)\, dW_\tau\right) &= \int_0^t (f(\tau))^2\, d\tau. \end{aligned}$$

Proof (1 of 2)

$$\begin{aligned}
\mathbb{E}\left(\int_0^t f(\tau)\, dW_\tau\right) &= \mathbb{E}\left(\lim_{n\to\infty}\sum_{k=1}^{n} f(t_{k-1})(W_{t_k} - W_{t_{k-1}})\right) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n} f(t_{k-1})\mathbb{E}\left(W_{t_k} - W_{t_{k-1}}\right) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n} f(t_{k-1})\cdot 0 \\
&= 0
\end{aligned}$$

Proof (2 of 2)

$$
\begin{aligned}
\mathrm{Var}\left(\int_0^t f(\tau)\,dW_\tau\right) &= \mathrm{Var}\left(\lim_{n\to\infty}\sum_{k=1}^{n} f(t_{k-1})(W_{t_k} - W_{t_{k-1}})\right) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}(f(t_{k-1}))^2 \mathrm{Var}\left(W_{t_k} - W_{t_{k-1}}\right) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}(f(t_{k-1}))^2 (t_k - t_{k-1}) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}(f(t_{k-1}))^2 \Delta t \\
&= \int_0^t (f(\tau))^2\,d\tau
\end{aligned}
$$

Example

If $f(t) = \sin t$, then find

$$\mathbb{E}\left(\int_0^t \sin\tau \, dW_\tau\right)$$

$$\mathrm{Var}\left(\int_0^t \sin\tau \, dW_\tau\right)$$

Example

If $f(t) = \sin t$, then find

$$\begin{aligned}\mathbb{E}\left(\int_0^t \sin\tau \, dW_\tau\right) &= 0 \\ \mathrm{Var}\left(\int_0^t \sin\tau \, dW_\tau\right)\end{aligned}$$

Example

If $f(t) = \sin t$, then find

$$
\begin{aligned}
\mathbb{E}\left(\int_0^t \sin\tau \, dW_\tau\right) &= 0 \\
\mathrm{Var}\left(\int_0^t \sin\tau \, dW_\tau\right) &= \int_0^t \sin^2\tau \, d\tau \\
&= \frac{1}{2}\int_0^t (1 - \cos 2\tau)\, d\tau \\
&= \frac{1}{2}t - \frac{1}{4}\sin 2t
\end{aligned}
$$

Riemann *vs.* Stochastic Integrals (1 of 5)

Suppose $f(t)$ is a continuously differentiable function on $[0, t]$ with $f(0) = 0$.

Then

$$\begin{aligned} \int_0^t f(\tau)\, df(\tau) &= \int_0^t f(\tau)\, f'(\tau)\, d\tau \\ &= \int_{f(0)}^{f(t)} u\, du \\ &= \frac{1}{2} u^2 \Big|_{f(0)}^{f(t)} \\ &= \frac{1}{2}(f(t))^2 \end{aligned}$$

Riemann *vs.* Stochastic Integrals (2 of 5)

Let W_t be a Wiener process and evaluate

$$\int_0^t W_\tau \, dW_\tau .$$

Riemann *vs.* Stochastic Integrals (2 of 5)

Let W_t be a Wiener process and evaluate

$$\int_0^t W_\tau \, dW_\tau .$$

Let $0 = t_0 < t_1 < \cdots < t_n = t$ be a partition of $[0, t]$.

$$\int_0^t W_\tau \, dW_\tau = \lim_{n \to \infty} \sum_{k=1}^{n} W_{t_{k-1}} \left(W_{t_k} - W_{t_{k-1}} \right)$$

Riemann *vs.* Stochastic Integrals (3 of 5)

$$
\begin{aligned}
&\int_0^t W_\tau \, dW_\tau \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} W_{t_{k-1}} \left(W_{t_k} - W_{t_{k-1}}\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(W_{t_{k-1}} W_{t_k} - W_{t_{k-1}}^2\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(-\frac{1}{2} W_{t_{k-1}}^2 + W_{t_{k-1}} W_{t_k} - \frac{1}{2} W_{t_{k-1}}^2\right) \\
&= \lim_{n\to\infty} \left(-\frac{1}{2} \sum_{k=1}^{n} W_{t_{k-1}}^2 + \sum_{k=1}^{n} W_{t_{k-1}} W_{t_k} - \frac{1}{2} \sum_{k=1}^{n} W_{t_{k-1}}^2\right)
\end{aligned}
$$

Riemann *vs.* Stochastic Integrals (4 of 5)

$$\int_0^t W_\tau \, dW_\tau$$

$$= \lim_{n\to\infty} \left[-\frac{1}{2} \sum_{k=1}^{n} W_{t_{k-1}}^2 + \sum_{k=1}^{n} W_{t_{k-1}} W_{t_k} - \frac{1}{2} \sum_{k=1}^{n} W_{t_{k-1}}^2 \right]$$

$$= \frac{1}{2} W_t^2 - \frac{1}{2} \lim_{n\to\infty} \left[\sum_{k=2}^{n+1} W_{t_{k-1}}^2 - 2 \sum_{k=1}^{n} W_{t_{k-1}} W_{t_k} + \sum_{k=1}^{n} W_{t_{k-1}}^2 \right]$$

$$= \frac{1}{2} W_t^2 - \frac{1}{2} \lim_{n\to\infty} \left[\sum_{k=1}^{n} W_{t_k}^2 - 2 \sum_{k=1}^{n} W_{t_{k-1}} W_{t_k} + \sum_{k=1}^{n} W_{t_{k-1}}^2 \right]$$

$$= \frac{1}{2} W_t^2 - \frac{1}{2} \lim_{n\to\infty} \sum_{k=1}^{n} \left(W_{t_k} - W_{t_{k-1}} \right)^2$$

Riemann *vs.* Stochastic Integrals (5 of 5)

$$\begin{aligned}
\int_0^t W_\tau \, dW_\tau &= \frac{1}{2} W_t^2 - \frac{1}{2} \lim_{n \to \infty} \sum_{k=1}^{n} \left(W_{t_k} - W_{t_{k-1}} \right)^2 \\
&= \frac{1}{2} W_t^2 - \frac{1}{2} \lim_{n \to \infty} \sum_{k=1}^{n} (t_k - t_{k-1}) \\
&= \frac{1}{2} W_t^2 - \frac{1}{2} t
\end{aligned}$$

Remarks:

- The stochastic integral of the Wiener process possesses an extra term ($-t/2$) which is not present in the Riemann integral.
- Integration by substitution (chain rule for derivatives) must operate differently for stochastic integrals.

ODE: Exponential Growth

Consider the familiar mathematical model for exponential growth of quantity P expressed in the form of an initial value problem:

$$\begin{aligned} \frac{dP}{dt} &= \mu P \\ P(0) &= P_0. \end{aligned}$$

If μ is a constant then $P(t) = P_0 e^{\mu t}$.

ODE: Exponential Growth

Consider the familiar mathematical model for exponential growth of quantity P expressed in the form of an initial value problem:

$$\begin{aligned} \frac{dP}{dt} &= \mu P \\ P(0) &= P_0. \end{aligned}$$

If μ is a constant then $P(t) = P_0 e^{\mu t}$.

The ODE above can be written in the equivalent form:

$$\frac{dP}{P} = \mu \, dt.$$

If we let $Z = \ln P$ then the ODE becomes

$$dZ = \mu \, dt.$$

Stochastic Differential Equation (SDE)

- Suppose the deterministic model is disturbed by a random influence.
- Interpret dW_t as random "noise" with a mean of 0 and variance dt.
- Perturb dZ by adding a random process with mean zero and variance $\sigma^2\,dt$.

$$dZ = \mu\,dt + \sigma\,dW_t$$

Stochastic Differential Equation (SDE)

- Suppose the deterministic model is disturbed by a random influence.
- Interpret dW_t as random “noise” with a mean of 0 and variance dt.
- Perturb dZ by adding a random process with mean zero and variance $\sigma^2\,dt$.

$$dZ = \mu\,dt + \sigma\,dW_t$$

This is mathematical model is an example of a **stochastic differential equation** of the type called a **generalized Wiener process.** The constant μ is called the **drift** and the constant σ is called the **volatility.** The solution to the SDE is

$$Z_t = Z_0 + \mu t + \int_0^t \sigma\,dW_\tau = Z_0 + \mu t + \sigma W_t.$$

Expectation and Variance

$$Z_t = Z_0 + \mu t + \int_0^t \sigma \, dW_\tau = Z_0 + \mu t + \sigma W_t.$$

The expression $\{Z_t\}_{t \geq 0}$ is a random process for which Z_t is normally distributed with

$$\begin{aligned} \mathbb{E}(Z_t - Z_0) &= \mu t \\ \mathrm{Var}(Z_t - Z_0) &= \sigma^2 t. \end{aligned}$$

Simple Generalization

If the drift and volatility are functions of t then

$$dZ = \mu(t)\,dt + \sigma(t)\,dW_t.$$

and by apply the appropriate integral (Riemann or stochastic, as needed) we have

$$Z_t = Z_0 + \int_0^t \mu(\tau)\,d\tau + \int_0^t \sigma(\tau)\,dW_\tau.$$

Itô Processes

A stochastic process of the form

$$dS = a(S, t)\, dt + b(S, t)\, dW_t$$

is called an **Itô process.**

Itô Processes

A stochastic process of the form

$$dS = a(S,t)\,dt + b(S,t)\,dW_t$$

is called an **Itô process.**

Suppose $Z = \ln S$, then $dZ = dS/S$ (by the chain rule).

- Are the following two stochastic processes equivalent?

$$\begin{aligned} dS &= \mu S\,dt + \sigma S\,dW_t \\ dZ &= \mu\,dt + \sigma\,dW_t \end{aligned}$$

- Which equation is the better model for the price of a security?

Discussion

$$
\begin{aligned}
dS &= \mu S \, dt + \sigma S \, dW_t \\
dZ &= \mu \, dt + \sigma \, dW_t
\end{aligned}
$$

- As $S \to 0^+$ then $\mu S \to 0$ and $\sigma S \to 0$.
- First equation makes a suitable mathematical model for a stock price $S \geq 0$, in second equation Z could go negative.
- Second equation can be integrated, first cannot.
- The two equations are not equivalent because the chain rule does not apply to functions of stochastic quantities.

Itô's Lemma

Lemma (Itô's Lemma)

Suppose that the random variable X is described by the Itô process

$$dX = a(X,t)\,dt + b(X,t)\,dW_t$$

where dW_t is the differential of the standard Wiener process. Suppose the random variable $Y = F(X,t)$. Then Y is described by the following Itô process.

$$dY = \left(a(X,t)F_X + F_t + \frac{1}{2}(b(X,t))^2 F_{XX}\right) dt + b(X,t)F_X dW_t$$

Taylor's Theorem

If $f(x)$ is an $(n+1)$-times differentiable function on an open interval containing x_0 then the function may be written as

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(\theta)}{(n+1)!}(x - x_0)^{n+1} \qquad (1)$$

The quantity θ lies between x and x_0. The last term above is usually called the Taylor remainder formula and is denoted by $R_{n+1}(x)$. The other terms form a polynomial in x of degree at most n and can be used as an approximation for $f(x)$ in a neighborhood of x_0.

Multivariable Form of Taylor's Theorem (1 of 2)

Suppose the function $F(y, z)$ has partial derivatives up to order three on an open disk containing the point with coordinates (y_0, z_0). Define the function $f(x) = F(y_0 + x\,h, z_0 + x\,k)$ where h and k are chosen small enough that $(y_0 + h, z_0 + k)$ lies within the disk surrounding (y_0, z_0). Since f is a function of the single variable x then we can use the single-variable form of Taylor's formula in Eq. (1) with $x_0 = 0$ and $x = 1$ to write

$$f(1) = f(0) + f'(0) + \frac{1}{2}f''(0) + R_3(1). \tag{2}$$

Using the multivariable chain rule for derivatives we have, upon differentiating $f(x)$ and setting $x = 0$,

$$\begin{aligned} f'(0) &= hF_y(y_0, z_0) + kF_z(y_0, z_0) && (3)\\ f''(0) &= h^2 F_{yy}(y_0, z_0) + 2hkF_{yz}(y_0, z_0) + k^2 F_{zz}(y_0, z_0). && (4) \end{aligned}$$

Multivariable Form of Taylor's Theorem (2 of 2)

We have made use of the fact that $F_{yz} = F_{zy}$ for this function under the smoothness assumptions. The remainder term R_3 contains only third-order partial derivatives of F evaluated somewhere on the line connecting the points (y_0, z_0) and $(y_0 + h, z_0 + k)$. Thus if we substitute Eqs. (3) and (4) into Eq. (2) we obtain

$$\begin{aligned} \Delta F &= f(1) - f(0) \qquad (5) \\ &= F(y_0 + h, z_0 + k) - F(y_0, z_0) \\ &= R_3(1) + hF_y(y_0, z_0) + kF_z(y_0, z_0) \\ &\quad + \frac{1}{2}\left(h^2 F_{yy}(y_0, z_0) + 2hkF_{yz}(y_0, z_0) + k^2 F_{zz}(y_0, z_0)\right). \end{aligned}$$

This last equation can be used to derive Itô's Lemma.

Proof (1 of 3)

Let X be a random variable described by an Itô process of the form

$$dX = a(X,t)\,dt + b(X,t)\,dW_t$$

where a and b are functions of X and t. Let $Y = F(X,t)$ be another random variable defined as a function of X and t. Given the Itô process which describes X we will now determine the Itô process which describes Y.

Proof (2 of 3)

Using a Taylor series expansion for Y detailed in Eq. (5) we find

$$\begin{aligned}\Delta Y &= F_X \Delta X + F_t \Delta t + \frac{1}{2} F_{XX} (\Delta X)^2 + F_{Xt} \Delta X \Delta t \\ &\quad + \frac{1}{2} F_{tt} (\Delta t)^2 + R_3 \\ &= F_X (a\Delta t + b\,dW_t) + F_t \Delta t + \frac{1}{2} F_{XX} (a\Delta t + b\,dW_t)^2 \\ &\quad + F_{Xt} (a\Delta t + b\,dW_t)\Delta t + \frac{1}{2} F_{tt} (\Delta t)^2 + R_3.\end{aligned}$$

Expand these products and discard any terms containing $(\Delta t)^n$ with $n > 1$.

Proof (3 of 3)

Upon simplifying, the expression ΔX has been replaced by the discrete version of the Itô process. Thus as Δt becomes small

$$\Delta Y \approx F_X(a\,dt + b\,dW_t) + F_t\,dt + \frac{1}{2!}F_{XX}b^2(dW_t)^2.$$

Using the relationship $(dW_t)^2 = dt$

$$\begin{aligned}\Delta Y &\approx F_X(a\,dt + b\,dW_t) + F_t\,dt + \frac{1}{2!}F_{XX}b^2\,dt \\ &= (a\,F_X + F_t + \frac{1}{2}b^2F_{XX})\,dt + bF_X\,dW_t.\end{aligned}$$

Examples (1 of 2)

If $Z = \ln S$ and

$$dS = \mu S\, dt + \sigma S\, dW_t,$$

find the stochastic process followed by Z.

Examples (1 of 2)

If $Z = \ln S$ and

$$dS = \mu S\,dt + \sigma S\,dW_t,$$

find the stochastic process followed by Z.

If $Z = \ln S$ then

$$\begin{aligned} dZ &= \left(\mu S\left[\frac{1}{S}\right] + 0 + \frac{1}{2}\sigma^2 S^2\left[-\frac{1}{S^2}\right]\right) dt + \sigma S\left(\frac{1}{S}\right) dW_t \\ &= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\,dW_t \end{aligned}$$

Examples (2 of 2)

If $S = e^Z$ and

$$dZ = \mu\, dt + \sigma\, dW_t,$$

find the stochastic process followed by S.

Examples (2 of 2)

If $S = e^Z$ and

$$dZ = \mu\, dt + \sigma\, dW_t,$$

find the stochastic process followed by S.

If $S = e^Z$ then

$$\begin{aligned} dS &= \left(\mu\left[e^Z\right] + 0 + \frac{1}{2}\sigma^2\left[e^Z\right]\right) dt + \sigma\left(e^Z\right) dW_t \\ &= \left(\mu + \frac{\sigma^2}{2}\right) S\, dt + \sigma S\, dW_t \end{aligned}$$

Example: Langevin Equation

Consider the stochastic differential equation

$$\begin{aligned} dX_t &= \mu X_t \, dt + \sigma \, dW_t \\ X(0) &= X_0. \end{aligned}$$

- Solve the initial value problem using Itô's lemma and the change of variables $F(X, t) = e^{-\mu t} X_t$.
- Find the mean and variance of X_t.

Solution (1 of 2)

Note that

$$\begin{aligned} F_X &= e^{-\mu t} \\ F_t &= -\mu e^{-\mu t} X \\ F_{XX} &= 0 \end{aligned}$$

which implies

$$\begin{aligned} dF &= \left(\mu F - \mu F + \frac{1}{2}\sigma^2(0) \right) dt + \sigma e^{-\mu t}\, dW_t \\ &= \sigma e^{-\mu t}\, dW_t \\ F(0) &= X_0. \end{aligned}$$

Integrating and using the initial condition yields

$$\begin{aligned} F(t) &= X_0 + \int_0^t \sigma e^{-\mu s}\, dW_s \\ X_t &= X_0 e^{\mu t} + \int_0^t \sigma e^{\mu(t-s)}\, dW_s. \end{aligned}$$

Solution (2 of 2)

Recall that

$$X_t = X_0 e^{\mu t} + \int_0^t \sigma e^{\mu(t-s)} \, dW_s.$$

The mean and variance are respectively

$$\begin{aligned} \mathbb{E}(X_t) &= X_0 e^{\mu t} \\ \mathrm{Var}(X_t) &= \int_0^t \left(\sigma e^{\mu(t-s)}\right)^2 dt = \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right). \end{aligned}$$

Homework

- Read Sections 5.4–5.8
- Exercises: 13–16, 18, 20

Credits

These slides are adapted from the textbook,

An Undergraduate Introduction to Financial Mathematics, 3rd edition, (2012).

author: J. Robert Buchanan

publisher: World Scientific Publishing Co. Pte. Ltd.

address: 27 Warren St., Suite 401–402, Hackensack, NJ 07601

ISBN: 978-9814407441