1. What is the lower bound for the price of a four-month European call option on a security when the price of the security is $15, the strike price is $12, and the interest rate is 6\% per year compounded continuously?

\[
C \geq S(0) - Ke^{-rT} \\
= 15 - 12e^{-(0.06/12)(4)} \\
\approx 3.23
\]

2. What is the lower bound for the price of a six-month call option on a security when the price of the security is $80, the strike price is $75, and the interest rate is 10\% per year compounded continuously?

\[
C \geq S(0) - Ke^{-rT} \\
= 80 - 75e^{-(0.10/12)(6)} \\
\approx 8.66
\]

3. Show that in the absence of arbitrage the price \( P \) of a European put option on a security of price \( S(0) \), with strike price \( K \), exercise time \( T \) when the interest rate is \( r \) per year compounded continuously obeys the relationship

\[
P \geq Ke^{-rT} - S(0).
\]

Suppose to the contrary that \( P < Ke^{-rT} - S(0) \). This is equivalent to the inequality,

\[
(S(0) + P)e^{rT} < K.
\]

Thus if an investor purchases the security and the put option with borrowed money and sells the stock at the exercise time, their net gain after paying back the loan will be at least \( K - (S(0) + P)e^{rT} > 0 \).

4. What is the lower bound for the price of a three-month European put option on a security when the price of the security is $38, the strike price is $40, and the interest rate is 10\% per year compounded continuously?

\[
P \geq Ke^{-rT} - S(0) \\
= 40e^{-(0.10/12)(3)} - 38 \\
\approx 1.01
\]

5. A one-month European put option on a security is selling for $2.50. The stock price is $47, the strike price is $50, and the interest rate is 6\% per year compounded continuously. What arbitrage opportunities exist?

Since the absence of arbitrage implies \( P \geq Ke^{-rT} - S(0) \) which is equivalent to the inequality

\[
S(0) + P \geq Ke^{-rT},
\]

we should compare the values of the quantities on both sides of the inequality.

\[
S(0) + P = 47 + 2.50 = 49.50 \\
Ke^{-rT} = 50e^{-(0.06/12)(1)} = 49.75
\]
We can see that $S(0) + P = 49.50 < 49.75 = Ke^{-rT}$ and thus an arbitrage opportunity exists. An investor can borrow funds to purchase the stock and the put option for $S(0) + P$. At the exercise time they will sell the stock for at least $K$. The net return will be $K - (S(0) + P)e^{rT} = 0.25 > 0$.

6. Put-call parity holds only for European options. Show that American options prices obey the following inequality,

$$S(0) - K \leq C - P \leq S(0) - Ke^{-rT}$$

where $S(0)$ is the current price of the security, $K$ is the strike price of both the American put and call options, $C$ is the price of the American call option, $P$ is the price of the American put option, $T$ is the exercise time, and $r$ is the interest rate per year compounded continuously.

To establish the first part of the inequality suppose to the contrary that $S(0) - K > C - P$ which is equivalent to $S(0) + P - C > K$. An investor could sell short the stock and the put option and buy the call option. The net proceeds from these transactions could be invested in a risk-free bond at interest rate $r$ compounded continuously. At any time $0 \leq t \leq T$, if the owner of the put option decides to exercise it, the investor uses the call option to buy back the stock. The bond now has value $(S(0) + P - C)e^{rt}$ and thus the net proceeds are

$$(S(0) + P - C)e^{rt} - K \geq S(0) + P - C - K > 0$$

by the initial assumption. If the owner of put allows it to expire unused then the investor buys the stock at time $T$ for amount $K$. In this case the net proceeds are

$$(S(0) + P - C)e^{rT} - K > 0.$$

Thus unless $S(0) - K \leq C - P$, an arbitrage opportunity exists.

Now suppose the second portion of the inequality is false, i.e., $C - P > S(0) - Ke^{-rT}$ which is equivalent to $K > (S(0) + P - C)e^{rT}$. An investor can borrow sufficient funds to buy the stock and the put and sell the call. At any time $0 \leq t \leq T$ the investor will owe $(S(0) + P - C)e^{rt}$ in principal and interest. If the owner of the call decides to exercise it at time $t$, the investor will sell the stock for $K$ and pay back the loan. The net return will be

$$K - (S(0) + P - C)e^{rt} \geq K - (S(0) + P - C)e^{rT} > 0.$$

If the owner of the call option never exercises it, the investor can sell the stock at time $T$, pay back the loan and will net proceeds of at least

$$K - (S(0) + P - C)e^{rT} > 0.$$

Thus unless $C - P \leq S(0) - Ke^{-rT}$, an arbitrage opportunity exists.

7. An American call option on a security with strike price $20$ and exercise time in five months is selling for $1.50$. Suppose the current price of the stock is $19$ and the interest rate is 10% per year compounded continuously. Find upper and lower bounds on the price of an American put option with the same strike price and exercise time.

$$S(0) - K \leq C - P \leq S(0) - Ke^{-rT}$$

where $S(0) = 19$, $K = 20$, $C = 1.50$, and $P$ is the price of the American put option. The interest rate is $r = 0.10/12 = 0.008333$ compounded continuously.

$$19 - 20 \leq 1.50 - P \leq 19 - 20e^{-0.008333(5)}$$

$$-1 \leq 1.50 - P \leq -0.1838$$

$$-2.50 \leq -P \leq -1.6838$$

$$1.68 \leq P \leq 2.50$$