The Concept of Limit
MATH 161 Calculus I

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Remark: Limits underlie most of the topics of calculus. It is important that you understand what limits mean, not just how to find them.

Today we will introduce

- numerical,
- graphical, and
- algebraic

approaches to finding the limit.
Suppose $f(x)$ is a function defined on an interval containing $a$ (but not necessarily at $a$).

\[
\lim_{x \to a^-} f(x) \quad \text{is read as “the limit of } f(x) \text{ as } x \text{ approaches } a \text{ from the left”}.
\]

\[
\lim_{x \to a^+} f(x) \quad \text{is read as “the limit of } f(x) \text{ as } x \text{ approaches } a \text{ from the right}.
\]

\[
\lim_{x \to a} f(x) \quad \text{is read as “the limit of } f(x) \text{ as } x \text{ approaches } a\text{”}.
\]
Notation

Suppose $f(x)$ is a function defined on an interval containing $a$ (but not necessarily at $a$).

\[\lim_{x \to a^-} f(x)\] is read as “the limit of $f(x)$ as $x$ approaches $a$ from the left”.

\[\lim_{x \to a^+} f(x)\] is read as “the limit of $f(x)$ as $x$ approaches $a$ from the right.

\[\lim_{x \to a} f(x)\] is read as “the limit of $f(x)$ as $x$ approaches $a$”.

**Remark:** the first two are called **one-sided limits**.
Example: Graphical Approach

$$\lim_{x \to -1} \frac{x^2 + x}{x^2 - x - 2}$$
Example: Numerical Approach

$$\lim_{x \to -1} \frac{x^2 + x}{x^2 - x - 2}$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.9</td>
<td>0.310345</td>
</tr>
<tr>
<td>-0.99</td>
<td>0.331104</td>
</tr>
<tr>
<td>-0.999</td>
<td>0.333111</td>
</tr>
<tr>
<td>-0.9999</td>
<td>0.333311</td>
</tr>
<tr>
<td>-1.0001</td>
<td>0.333356</td>
</tr>
<tr>
<td>-1.001</td>
<td>0.333555</td>
</tr>
<tr>
<td>-1.01</td>
<td>0.335548</td>
</tr>
<tr>
<td>-1.1</td>
<td>0.354839</td>
</tr>
</tbody>
</table>
Example: Algebraic Approach

\[
\lim_{{x \to -1}} \frac{x^2 + x}{x^2 - x - 2} = \lim_{{x \to -1}} \frac{x(x + 1)}{(x - 2)(x + 1)} \\
= \lim_{{x \to -1}} \frac{x}{x - 2} \\
= \frac{1}{3}
\]
Example: Graphical Approach

\[
\lim_{x \to 2} \frac{x^2 + x}{x^2 - x - 2}
\]

Conclusion: \( \lim_{x \to 2} \frac{x^2 + x}{x^2 - x - 2} \) does not exist.
Example: Graphical Approach

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<td>21.</td>
</tr>
<tr>
<td>2.01</td>
<td>201.</td>
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<tr>
<td>2.001</td>
<td>2001.</td>
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<tr>
<td>2.0001</td>
<td>20001.</td>
</tr>
<tr>
<td>1.9999</td>
<td>−19999.</td>
</tr>
<tr>
<td>1.999</td>
<td>−1999.</td>
</tr>
<tr>
<td>1.99</td>
<td>−199.</td>
</tr>
<tr>
<td>1.9</td>
<td>−19.</td>
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</table>
Example: Numerical Approach

\[ \lim_{x \to 2} \frac{x^2 + x}{x^2 - x - 2} \]

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**Conclusion:** \( \lim_{x \to 2} \frac{x^2 + x}{x^2 - x - 2} \) does not exist.
Example: Limit from the Left

\[
\lim_{{x \to 0^-}} \frac{|x|}{{-x}}
\]

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</tr>
<tr>
<td>-0.0001</td>
<td>-1.0</td>
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</table>
Example: Limit from the Right

\[ \lim_{x \to 0^+} \frac{|x|}{x} \]

\[ x \quad f(x) \]
\[ 0.1 \quad 1. \]
\[ 0.01 \quad 1. \]
\[ 0.001 \quad 1. \]
\[ 0.0001 \quad 1. \]
What About $\lim_{x \to 0} \frac{|x|}{x}$?

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \neq 1 = \lim_{x \to 0^+} \frac{|x|}{x}$$

so even though the one-sided limits exist,

$$\lim_{x \to 0} \frac{|x|}{x}$$ does not exist.

Remark: A limit exists if and only if both corresponding one-sided limits exist and are equal.
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$$\lim_{x \to 0} \frac{|x|}{x}$$

does not exist.

**Remark:** a limit exists if and only if both corresponding one-sided limits exist and are equal.
Examples

Use graphical and/or numerical evidence to conjecture the following limits.

1. \( \lim_{x \to -1} \frac{x^2 - 1}{x + 1} \)
2. \( \lim_{x \to \pi^-} \frac{\sin x}{x - \pi} \)
3. \( \lim_{x \to 0} x \sin \left( \frac{1}{x} \right) \)
4. \( \lim_{x \to 0^+} \frac{-1 + \cos x}{x} \)
Homework

- Read Section 1.2
- Exercises: 1–31 odd