Increasing and Decreasing Functions

MATH 161 Calculus I

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- when a local extremum occurs, it occurs at a critical number \((f'(c) = 0 \text{ or } f'(c) \text{ is undefined})\), but
- not every critical number is a local extremum.
Example

If \( f(x) = x^3 \), \( x = 0 \) is a critical number but \( f(0) = 0 \) is neither a local minimum nor local maximum.
Increasing and Decreasing Functions

To classify critical numbers as local extrema we need the following concepts:

**Definition**

A function $f$ is **strictly increasing** on an interval $I$ if for every $x_1, x_2 \in I$ with $x_1 < x_2$ we have $f(x_1) < f(x_2)$ [in other words, $f(x)$ gets larger as $x$ gets larger].

A function $f$ is **strictly decreasing** on an interval $I$ if for every $x_1, x_2 \in I$ with $x_1 < x_2$ we have $f(x_1) > f(x_2)$ [in other words, $f(x)$ gets smaller as $x$ gets larger].
Increasing/Decreasing and Derivative (1 of 2)

Consider the graph of $f$ below and think about the relationship between the derivative of $f$ and the increasing and decreasing behavior of $f$. 
Theorem
Suppose that $f$ is differentiable on an interval $I$.

1. If $f'(x) > 0$ for all $x \in I$ then $f$ is increasing on $I$.
2. If $f'(x) < 0$ for all $x \in I$ then $f$ is decreasing on $I$.

Proof.
Let $x_1, x_2 \in I$ with $x_1 < x_2$ and use the MVT.
Examples

Find the intervals where the following functions are increasing and decreasing.

- $f(x) = x^3 + 2x^2 + 3$
- $g(x) = \ln(x^2 - 4)$
- $h(x) = \frac{x}{1 + x^4}$

Use the increasing/decreasing information to sketch graphs of the functions.
Solution: \( f(x) = x^3 + 2x^2 + 3 \)

\[
f'(x) = 3x^2 + 4x
\]

\[
0 = x(3x + 4)
\]

\[
x = 0 \quad \text{or} \quad x = -\frac{4}{3}
\]

Function \( f(x) \) is decreasing on \((-4/3, 0)\) and increasing on \((-\infty, -4/3) \cup (0, \infty)\).
Solution: $g(x) = \ln(x^2 - 4)$

The domain of $g(x)$ is $(-\infty, -2) \cup (2, \infty)$.

$$g'(x) = \frac{2x}{x^2 - 4}$$

Function $g(x)$ is decreasing on $(-\infty, -2)$ and increasing on $(2, \infty)$. 
Solution: \( h(x) = \frac{x}{1+x^4} \)

\[
h'(x) = \frac{1 - 3x^4}{(1 + x^4)^2}
\]

\[
0 = 1 - 3x^4
\]

\[
x = \pm \frac{1}{\sqrt[4]{3}} \approx 0.759836
\]

Function \( h(x) \) is increasing on \((-1/\sqrt[4]{3}, 1/\sqrt[4]{3})\) and decreasing on \((-\infty, -1/\sqrt[4]{3}) \cup (1/\sqrt[4]{3}, \infty)\).
First Derivative Test

Theorem (First Derivative Test)

Suppose that $f$ is continuous on interval $[a, b]$ and $c \in (a, b)$ is a critical number.

1. If $f'(x) > 0$ for all $x \in (a, c)$ and $f'(x) < 0$ for all $x \in (c, b)$ [in other words $f$ changes from increasing to decreasing at $c$] then $f(c)$ is a local maximum.

2. If $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all $x \in (c, b)$ [in other words $f$ changes from decreasing to increasing at $c$] then $f(c)$ is a local minimum.

3. If $f'(x)$ has the same sign on $(a, c)$ and $(c, b)$, then $f(c)$ is not a local extremum.
Examples

Find the local extrema (if any) for the following functions.

- $f(x) = x^2 e^{-x}$

- $g(x) = x^{4/3} + 4x^{1/3}$

- $h(x) = \frac{x^2 + 2}{(1 + x)^2}$
Examples

Find the local extrema (if any) for the following functions.

▶ \( f(x) = x^2 e^{-x} \)

\[ f'(x) = x(2 - x)e^{-x} \]

▶ \( g(x) = x^{4/3} + 4x^{1/3} \)

▶ \( h(x) = \frac{x^2 + 2}{(1 + x)^2} \)
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  $f'(x) = x(2 - x)e^{-x}$

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  $g'(x) = \frac{4(x + 1)}{3x^{2/3}}$

- $h(x) = \frac{x^2 + 2}{(1 + x)^2}$

Examples

Find the local extrema (if any) for the following functions.

- \( f(x) = x^2 e^{-x} \)
  \[ f'(x) = x(2 - x)e^{-x} \]

- \( g(x) = x^{4/3} + 4x^{1/3} \)
  \[ g'(x) = \frac{4(x + 1)}{3x^{2/3}} \]

- \( h(x) = \frac{x^2 + 2}{(1 + x)^2} \)
  \[ h'(x) = \frac{2(x - 2)}{(x + 1)^3} \]
The derivative:

\[ f'(x) = x(2 - x)e^{-x} \]

Sign diagram for \( f'(x) \):

\[ f'(x) \]

\[ \begin{array}{c|cccccc|ccc} & - & - & - & + & + & + & + & + & - & - & - \\ \hline -1 & 0 & 1 & 2 & 3 & x \end{array} \]

- Local minimum: \((0, f(0)) = (0, 0)\)
- Local maximum: \((2, f(2)) = (2, 4e^{-2})\)
\[ f(x) = x^2 e^{-x} \]
The derivative:

\[ g'(x) = \frac{4(x + 1)}{3x^{2/3}} \]

Sign diagram for \( g'(x) \):

Local minimum: \((-1, g(-1)) = (-1, -3)\)

Local maximum: none
\( g(x) = x^{4/3} + 4x^{1/3} \)
\[ h(x) = \frac{x^2 + 2}{(1+x)^2} \] (2 of 2)

The derivative:

\[ h'(x) = \frac{2(x - 2)}{(x + 1)^3} \]

Sign diagram for \( h'(x) \):

- Local minimum: \((2, h(2)) = (2, 2/3)\)
- Local maximum: none
\[ h(x) = \frac{x^2 + 2}{(1+x)^2} \]
Homework

- Read Section 3.4
- Exercises: 1–39 odd