Sums and Sigma Notation

MATH 161 Calculus I

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Overview

- We have mentioned that the antiderivative is associated with the process of accumulation or summing.
- Today we will introduce the notation and properties of summations.
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\[ \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n \]

- \( i \): summation index
- \( a_i \): summand
Examples

\[
\sum_{i=1}^{5} \pi \\
\sum_{i=1}^{4} i \\
\sum_{i=1}^{6} \frac{i}{2}
\]
Examples

\[ \sum_{i=1}^{5} \pi = \pi + \pi + \pi + \pi + \pi = 5\pi \]

\[ \sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10 \]

\[ \sum_{i=1}^{6} \frac{i}{2} = \frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \frac{4}{2} + \frac{5}{2} + \frac{6}{2} = \frac{21}{2} \]
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Examples

Write the following sums in summation notation.

▶ $4 + 8 + 12 + \cdots + 48$

▶ $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{100}$

▶ $e^{\pi/4} + e^{\pi/2} + e^{3\pi/4} + \cdots + e^{2\pi}$
Examples

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▸ $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{100} = \sum_{k=1}^{100} \frac{1}{k}$

▸ $e^{\pi/4} + e^{\pi/2} + e^{3\pi/4} + \cdots + e^{2\pi} = \sum_{i=1}^{8} e^{i\pi/4}$
Theorem

If $n$ is any positive integer and $c$ is any constant, then

- $\sum_{i=1}^{n} c = cn$ \textit{(sum of a constant)},

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ \textit{(sum of first $n$ positive integers)},

- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ \textit{(sum of the first $n$ squares)}. 
Summation Formulas (2 of 2)

Proof.

\[
\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + (n - 1) + n = S
\]

\[
\begin{array}{cccccc}
1 & 2 & 3 & \cdots & n-1 & n \\
n & n-1 & n-2 & \cdots & 2 & 1 \\
n+1 & n+1 & n+1 & \cdots & n+1 & n+1 \\
\end{array}
\]

\[
2S = n(n+1) = 2 \sum_{i=1}^{n} i
\]

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}
\]
A Property of Summations

Theorem

For any constants $c$ and $d$,

$$\sum_{i=1}^{n} (c \ a_i + d \ b_i) = c \sum_{i=1}^{n} a_i + d \sum_{i=1}^{n} b_i.$$
Examples

Use summation formulas and properties of sums to evaluate the following summations.

1. \[ \sum_{i=1}^{40} (3i - 5) = 2260 \]
2. \[ \sum_{i=1}^{145} (i^2 + 4i - 2) = 145,068,795 \]
3. \[ \sum_{i=1}^{n} \frac{1}{n} \left[ \left( \frac{i}{n} \right)^2 - 3 \left( \frac{i}{n} \right) \right] \]
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\[ \sum_{i=1}^{n} \frac{1}{n} \left[ \left( \frac{i}{n} \right)^2 - 3 \left( \frac{i}{n} \right) \right] = \frac{1 - 6n - 7n^2}{6n^2} \]
Example

Let \( f(x) = x^2 + 3x + 4 \) and use your calculator to

- find the sum of the values of \( f(x) \) for \( x = 2.05, x = 2.10, \ x = 2.15, \ldots, x = 3.00, \)

- find the sum \( \sum_{i=1}^{n} f(x_i) \Delta x \), where \( \Delta x \) is the difference
between consecutive values of \( x \).
Example
Let \( f(x) = x^2 + 3x + 4 \) and use your calculator to

\[ \sum_{i=1}^{20} f(2 + 0.05i) = \sum_{i=1}^{20} \left[ (2 + 0.05i)^2 + 3(2 + 0.05i) + 4 \right] = 360.675 \]

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\sum_{i=1}^{20} f(2 + 0.05i) \Delta x
\]

\[
= (0.05) \sum_{i=1}^{20} \left[ (2 + 0.05i)^2 + 3(2 + 0.05i) + 4 \right] = 18.0338
\]
Suppose the velocity of a falling object is recorded at various times.

<table>
<thead>
<tr>
<th>$t$ (s)</th>
<th>0.0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v$ (m/s)</td>
<td>10.0</td>
<td>14.9</td>
<td>19.8</td>
<td>24.7</td>
<td>29.6</td>
<td>34.5</td>
<td>39.4</td>
</tr>
</tbody>
</table>

Estimate the distance the object falls over the interval $0 \leq t \leq 3$. 
Solution

- Distance equals velocity multiplied by time.
- We will average the velocities measured at the beginning and end of each interval (for example, the average velocity during the first interval is \((10.0 + 14.9)/2 = 12.45 \text{ m/s})\).

\[
d = (12.45)(0.5) + (17.35)(0.5) + (22.25)(0.5) + (27.15)(0.5) \\
+ (32.05)(0.5) + (36.95)(0.5) \\
= 0.5(12.45 + 17.35 + 22.25 + 27.15 + 32.05 + 36.95) \\
= 0.5(148.2) \\
= 74.1 \text{ m}
\]
Homework

- Read Section 4.2
- Exercises: 1–31 odd