Given a pair a parametric equations

\[ x = f(t) \]
\[ y = g(t) \]

for \( a \leq t \leq b \) we know how to graph the parametric curve.

Today we will focus our attention on finding the slope of the tangent line to the graph and the area enclosed by a simple closed curve.
Suppose $x = f(t)$ and $y = g(t)$, by the Chain Rule for Derivatives

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}.$$

Let $(x_0, y_0) = (x(t_0), y(t_0))$ then so long as $\frac{dx}{dt}(t_0) \neq 0$ then

$$\frac{dy}{dx}(x_0) = \frac{dy}{dt}(t_0) \frac{dx}{dt}(t_0).$$

**Remark:** If $x'(t_0) = y'(t_0) = 0$ then

$$\frac{dy}{dx}(x_0) = \lim_{t \to t_0} \frac{dy}{dt} \frac{dx}{dt} = \lim_{t \to t_0} \frac{y'(t)}{x'(t)},$$

provided the limit exists.
Example

Find the slope and equation of the tangent line for the following parametric equations at $t = 1$.

$$x = t^3 - t$$
$$y = t^4 - 5t^2 + 4$$
Solution

\[
\frac{dx}{dt} = 3t^2 - 1 \\
\frac{dy}{dt} = 4t^3 - 10t \\
\left.\frac{dy}{dx}\right|_{t=1} = \frac{4 - 10}{3 - 1} = -3
\]

Since \((x(1), y(1)) = (0, 0)\) then the equation of the tangent line is

\[y = -3x\]
The second derivative is the derivative of the first derivative.

\[
\frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{dy}{dx} \right) \frac{dx}{dt}
\]
Find the Second Derivative (Concavity)

The second derivative is the derivative of the first derivative.

\[
\frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{dy}{dx} \right)
\]

Note: \( \frac{d^2 y}{dx^2} \neq \frac{d^2 y}{dt^2} \)
Example

Find $\frac{d^2y}{dx^2}$ for the following parametric equations at $t = 1$.

\[ x = t^3 - t \]
\[ y = t^4 - 5t^2 + 4 \]
\[
\frac{d^2 y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) \frac{dx}{dt} = \frac{\frac{d}{dt} \left( \frac{4t^3 - 10t}{3t^2 - 1} \right)}{3t^2 - 1}
\]

\[
= \frac{(12t^2 - 10)(3t^2 - 1) - (4t^3 - 10t)(6t)}{(3t^2 - 1)^2} \frac{1}{3t^2 - 1}
\]

\[
\left. \frac{d^2 y}{dx^2} \right|_{t=1} = \frac{(12 - 10)(3 - 1) - (4 - 10t)(6)}{(3 - 1)^2} = 5
\]
Theorem

Suppose that $x'(t)$ and $y'(t)$ are continuous. Then for the curve defined by the parametric equations

\[
x = x(t) \\
y = y(t)
\]

1. If $y'(c) = 0$ and $x'(c) \neq 0$, there is a horizontal tangent line at the point $(x(c), y(c))$.
2. If $x'(c) = 0$ and $y'(c) \neq 0$, there is a vertical tangent line at the point $(x(c), y(c))$. 
Find the points at which the graph of the following parametric equations has horizontal or vertical tangent lines.

\begin{align*}
x &= t^2 - 1 \\
y &= t^4 - 4t^2
\end{align*}
Solution

\[
\frac{dx}{dt} = 2t
\]
\[
\frac{dy}{dt} = 4t^3 - 8t = 4t(t^2 - 2)
\]
\[ \frac{dx}{dt} = 2t \]
\[ \frac{dy}{dt} = 4t^3 - 8t = 4t(t^2 - 2) \]

Since when \( y'(\pm\sqrt{2}) = 0 \) and \( x'(\pm\sqrt{2}) = \pm2\sqrt{2} \neq 0 \) then the graph has horizontal tangents when \( t = \pm\sqrt{2} \).

\[ \left( x(\pm\sqrt{2}), y(\pm\sqrt{2}) \right) = (2 - 1, 4 - 8) = (1, -4) \]
Solution

\[\frac{dx}{dt} = 2t\]
\[\frac{dy}{dt} = 4t^3 - 8t = 4t(t^2 - 2)\]

Since when \(y'(\pm \sqrt{2}) = 0\) and \(x'(\pm \sqrt{2}) = \pm 2\sqrt{2} \neq 0\) then the graph has horizontal tangents when \(t = \pm \sqrt{2}\).

\[\left( x(\pm \sqrt{2}), y(\pm \sqrt{2}) \right) = (2 - 1, 4 - 8) = (1, -4)\]

Note that \(x'(0) = y'(0) = 0\) so the slope of the tangent line when \(t = 0\) is

\[\lim_{t \to 0} \frac{4t^3 - 8t}{2t} = \lim_{t \to 0} 2(t^2 - 2) = -4 \neq 0.\]

There are no vertical tangents.
If the position of a moving object is given by the parametric equations

\[
\begin{align*}
  x &= x(t) \\
  y &= y(t)
\end{align*}
\]

where \( x(t) \) and \( y(t) \) are differentiable we say

- the **horizontal component of velocity** is given by \( x'(t) \),
- the **vertical component of velocity** is given by \( y'(t) \), and
- the **speed** is given by \( \sqrt{[x'(t)]^2 + [y'(t)]^2} \).
Find the components of velocity and the speed of an object moving according to the parametric equations

\[ x = 3 \cos t + \sin 3t \]
\[ y = 3 \sin t + \cos 3t \]

at \( t = \pi/2 \).
Solution

\[\begin{align*}
x'(t) &= -3 \sin t + 3 \cos 3t \\
y'(t) &= 3 \cos t - 3 \sin 3t
\end{align*}\]
Solution

\[ x'(t) = -3 \sin t + 3 \cos 3t \]
\[ y'(t) = 3 \cos t - 3 \sin 3t \]
\[ x'(\pi/2) = -3 \]
\[ y'(\pi/2) = 3 \]
Solution

\[ x'(t) = -3 \sin t + 3 \cos 3t \]
\[ y'(t) = 3 \cos t - 3 \sin 3t \]
\[ x'(\pi/2) = -3 \]
\[ y'(\pi/2) = 3 \]
\[ s(\pi/2) = \sqrt{(-3)^2 + (3)^2} = 3\sqrt{2} \]
Recall: if \( y = f(x) \geq 0 \) for \( a \leq x \leq b \) then the area under the curve, above the \( x \)-axis and between \( x = a \) and \( x = b \) is given by

\[
A = \int_{a}^{b} f(x) \, dx = \int_{a}^{b} y \, dx.
\]
**Recall:** if \( y = f(x) \geq 0 \) for \( a \leq x \leq b \) then the area under the curve, above the \( x \)-axis and between \( x = a \) and \( x = b \) is given by

\[
A = \int_a^b f(x) \, dx = \int_a^b y \, dx.
\]

If the region is enclosed by parametrically defined curves

\[
x = x(t) \\
y = y(t)
\]

with \( c \leq t \leq d \) then

\[
A = \int_a^b \frac{y}{y(t)} \frac{dx}{x'(t)} \, dt = \int_c^d y(t)x'(t) \, dt.
\]
Suppose that the parametric equations \( x = x(t) \) and \( y = y(t) \) with \( c \leq t \leq d \) describe a curve that is traced out clockwise exactly once as \( t \) increases from \( c \) to \( d \) and where the curve does not intersect itself, except that the initial and terminal points are the same, i.e., \( x(c) = x(d) \) and \( y(c) = y(d) \). Then the enclosed area is given by

\[
A = \int_{c}^{d} y(t)x'(t) \, dt = -\int_{c}^{d} x(t)y'(t) \, dt.
\]

If the curve is traced out counterclockwise, then the enclosed curve is given by

\[
A = -\int_{c}^{d} y(t)x'(t) \, dt = \int_{c}^{d} x(t)y'(t) \, dt.
\]
Find the area enclosed by the graph of the parametric curve described by

\[ x = t - \sin t \]
\[ y = 1 - \cos t \]

for \( 0 \leq t \leq 2\pi \).
Solution

\[ A = \int_0^{2\pi} (1 - \cos t)(1 - \cos t) \, dt \]

\[ = \int_0^{2\pi} (1 - 2 \cos t + \cos^2 t) \, dt \]

\[ = \int_0^{2\pi} \left( 1 - 2 \cos t + \frac{1}{2}(1 + \cos 2t) \right) \, dt \]

\[ = \int_0^{2\pi} \left( \frac{3}{2} - 2 \cos t + \frac{1}{2} \cos 2t \right) \, dt \]

\[ = 3\pi \]
The ellipse whose general formula is \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) for \( a, b > 0 \) is described parametrically by

\[
\begin{align*}
x &= a \cos t \\
y &= b \sin t
\end{align*}
\]

for \( 0 \leq t \leq 2\pi \). Use the parametric equations to find a formula for the area of an ellipse.
Solution

\[ A = - \int_{0}^{2\pi} (b \sin t) (-a \sin t) \, dt \]
\[ = ab \int_{0}^{2\pi} \sin^2 t \, dt \]
\[ = \frac{ab}{2} \int_{0}^{2\pi} (1 - \cos 2t) \, dt \]
\[ = ab \pi \]
Homework

- Read Section 9.2
- Exercises: 1–29 odd, 35