Calculus and Parametric Equations
MATH 211, *Calculus II*

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Spring 2018
Given a pair a parametric equations

\[ x = f(t) \]
\[ y = g(t) \]

for \( a \leq t \leq b \) we know how to graph the parametric curve.
Objectives

Today we will focus our attention on finding:

▶ the slope of the tangent line to the graph of a parametric curve,
▶ the area enclosed by a simple closed curve,
▶ the arc length of a parametric curve, and
▶ the surface area of a surface of revolution.
Slope of the Tangent Line

Suppose \( x = f(t) \) and \( y = g(t) \), by the Chain Rule for Derivatives

\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}.
\]

Let \((x_0, y_0) = (x(t_0), y(t_0))\) then so long as \( \frac{dx}{dt}(t_0) \neq 0 \) then

\[
\frac{dy}{dx}(x_0) = \frac{\frac{dy}{dt}(t_0)}{\frac{dx}{dt}(t_0)}.
\]

Remark: If \( x'(t_0) = y'(t_0) = 0 \) then

\[
\frac{dy}{dx}(x_0) = \lim_{t \to t_0} \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \lim_{t \to t_0} \frac{y'(t)}{x'(t)},
\]

provided the limit exists.
Example

Find the slope and equation of the tangent line for the following parametric equations at $t = 1$.

$$x = t^3 - t$$
$$y = t^4 - 5t^2 + 4$$
Solution

\[
\frac{dx}{dt} = 3t^2 - 1
\]
\[
\frac{dy}{dt} = 4t^3 - 10t
\]
\[
\left.\frac{dy}{dx}\right|_{t=1} = \frac{4 - 10}{3 - 1} = -3
\]

Since \((x(1), y(1)) = (0, 0)\) then the equation of the tangent line is

\[y = -3x\]
Find the Second Derivative (Concavity)

The second derivative is the derivative of the first derivative.

\[
\frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{dy}{dx} \right)
\]
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\]

Note: \( \frac{d^2y}{dx^2} \neq \frac{d^2y}{dt^2} \frac{dx}{dt^2} \)
Example

Find \( \frac{d^2y}{dx^2} \) for the following parametric equations at \( t = 1 \).

\[
\begin{align*}
  x &= t^3 - t \\
  y &= t^4 - 5t^2 + 4
\end{align*}
\]
\[ \frac{d^2 y}{dx^2} = \frac{\frac{d}{dt} \left( \frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt} \left( \frac{4t^3 - 10t}{3t^2 - 1} \right)}{3t^2 - 1} \]

\[ = \frac{(12t^2 - 10)(3t^2 - 1) - (4t^3 - 10t)(6t)}{(3t^2 - 1)^2} \]

\[ = \frac{3t^2 - 1}{(3t^2 - 1)^2} \]

\[ \left. \frac{d^2 y}{dx^2} \right|_{t=1} = \left. \frac{(12 - 10)(3 - 1) - (4 - 10)(6)}{(3 - 1)^2} \right| = 5 \]
Finding Horizontal and Vertical Tangent Lines

Theorem
Suppose that $x'(t)$ and $y'(t)$ are continuous. Then for the curve defined by the parametric equations

\[
\begin{align*}
x &= x(t) \\
y &= y(t)
\end{align*}
\]

1. if $y'(c) = 0$ and $x'(c) \neq 0$, there is a horizontal tangent line at the point $(x(c), y(c))$.
2. if $x'(c) = 0$ and $y'(c) \neq 0$, there is a vertical tangent line at the point $(x(c), y(c))$. 
Example

Find the points at which the graph of the following parametric equations has horizontal or vertical tangent lines.

\[ x = t^2 - 1 \]
\[ y = t^4 - 4t^2 \]
Solution

\[ \frac{dx}{dt} = 2t \]

\[ \frac{dy}{dt} = 4t^3 - 8t = 4t(t^2 - 2) \]
Solution

\[
\frac{dx}{dt} = 2t \\
\frac{dy}{dt} = 4t^3 - 8t = 4t(t^2 - 2)
\]

Since when \( y'(\pm \sqrt{2}) = 0 \) and \( x'(\pm \sqrt{2}) = \pm 2\sqrt{2} \neq 0 \) then the graph has horizontal tangents when \( t = \pm \sqrt{2} \).

\[
\left( x(\pm \sqrt{2}), y(\pm \sqrt{2}) \right) = (2 - 1, 4 - 8) = (1, -4)
\]
Solution

\[
\frac{dx}{dt} = 2t \\
\frac{dy}{dt} = 4t^3 - 8t = 4t(t^2 - 2)
\]

Since when \( y'(±\sqrt{2}) = 0 \) and \( x'(±\sqrt{2}) = ±2\sqrt{2} \neq 0 \) then the graph has horizontal tangents when \( t = ±\sqrt{2} \).

\[
\left(x(±\sqrt{2}), y(±\sqrt{2})\right) = (2 - 1, 4 - 8) = (1, -4)
\]

Note that \( x'(0) = y'(0) = 0 \) so the slope of the tangent line when \( t = 0 \) is

\[
\lim_{t \to 0} \frac{4t^3 - 8t}{2t} = \lim_{t \to 0} 2(t^2 - 2) = -4 \neq 0.
\]

There are no vertical tangents.
Velocity and Speed

If the position of a moving object is given by the parametric equations

\[ x = x(t) \]
\[ y = y(t) \]

where \( x(t) \) and \( y(t) \) are differentiable we say

- the **horizontal component of velocity** is given by \( x'(t) \),
- the **vertical component of velocity** is given by \( y'(t) \), and
- the **speed** is given by \( \sqrt{[x'(t)]^2 + [y'(t)]^2} \).
Example

Find the components of velocity and the speed of an object moving according to the parametric equations

\[ x = 3 \cos t + \sin 3t \]
\[ y = 3 \sin t + \cos 3t \]

at \( t = \pi/2 \).
Solution

\[ x'(t) = -3 \sin t + 3 \cos 3t \]
\[ y'(t) = 3 \cos t - 3 \sin 3t \]
Solution

\[ x'(t) = -3 \sin t + 3 \cos 3t \]
\[ y'(t) = 3 \cos t - 3 \sin 3t \]
\[ x'(\pi/2) = -3 \]
\[ y'(\pi/2) = 3 \]
Solution

\[ x'(t) = -3 \sin t + 3 \cos 3t \]
\[ y'(t) = 3 \cos t - 3 \sin 3t \]
\[ x'\left(\frac{\pi}{2}\right) = -3 \]
\[ y'\left(\frac{\pi}{2}\right) = 3 \]
\[ s\left(\frac{\pi}{2}\right) = \sqrt{(-3)^2 + (3)^2} = 3\sqrt{2} \]
Area Enclosed by a Curve (1 of 2)

Recall: if \( y = f(x) \geq 0 \) for \( a \leq x \leq b \) then the area under the curve, above the \( x \)-axis and between \( x = a \) and \( x = b \) is given by

\[
A = \int_a^b f(x) \, dx = \int_a^b y \, dx.
\]
Recall: if \( y = f(x) \geq 0 \) for \( a \leq x \leq b \) then the area under the curve, above the \( x \)-axis and between \( x = a \) and \( x = b \) is given by
\[
A = \int_a^b f(x) \, dx = \int_a^b y \, dx.
\]

If the region is enclosed by parametrically defined curves
\[
\begin{align*}
x &= x(t) \\
y &= y(t)
\end{align*}
\]
with \( c \leq t \leq d \) then
\[
A = \int_a^b \left( \int_{y(t)}^{y(t)} dx \right) dt = \int_c^d y(t) x'(t) \, dt.
\]
Theorem
Suppose that the parametric equations $x = x(t)$ and $y = y(t)$ with $c \leq t \leq d$ describe a curve that is traced out clockwise exactly once as $t$ increases from $c$ to $d$ and where the curve does not intersect itself, except that the initial and terminal points are the same, i.e., $x(c) = x(d)$ and $y(c) = y(d)$. Then the enclosed area is given by

$$A = \int_{c}^{d} y(t)x'(t) \, dt = -\int_{c}^{d} x(t)y'(t) \, dt.$$ 

If the curve is traced out counterclockwise, then the enclosed curve is given by

$$A = -\int_{c}^{d} y(t)x'(t) \, dt = \int_{c}^{d} x(t)y'(t) \, dt.$$
Example

Find the area enclosed by the graph of the parametric curve described by

\[ x = t - \sin t \]
\[ y = 1 - \cos t \]

for \( 0 \leq t \leq 2\pi \).
Solution

\[ A = \int_{0}^{2\pi} (1 - \cos t)(1 - \cos t) \, dt \]

\[ = \int_{0}^{2\pi} (1 - 2 \cos t + \cos^2 t) \, dt \]

\[ = \int_{0}^{2\pi} \left(1 - 2 \cos t + \frac{1}{2}(1 + \cos 2t)\right) \, dt \]

\[ = \int_{0}^{2\pi} \left(3 - 2 \cos t + \frac{1}{2} \cos 2t\right) \, dt \]

\[ = 3\pi \]
The ellipse whose general formula is \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) for \( a, b > 0 \) is described parametrically by

\[
\begin{align*}
  x &= a \cos t \\
  y &= b \sin t
\end{align*}
\]

for \( 0 \leq t \leq 2\pi \). Use the parametric equations to find a formula for the area of an ellipse.
Solution

\[
A = - \int_0^{2\pi} (b \sin t)(-a \sin t) \, dt
\]

\[
= ab \int_0^{2\pi} \sin^2 t \, dt
\]

\[
= \frac{ab}{2} \int_0^{2\pi} (1 - \cos 2t) \, dt
\]

\[
= ab \pi
\]
Homework

- Read Section 10.2
- Exercises: WebAssign/D2L