Integration by Parts
MATH 211, Calculus II

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If necessary refresh your skills with basic integration including \textbf{integration by substitution} by reviewing Section 6.1 and working the exercises at the end of the section.

Today’s discussion will focus on the second major technique of integration, \textbf{integration by parts}.

We will see that integration by parts is related to the product rule for derivatives.
Suppose that $u$ and $v$ are functions of $x$ then

\[
\frac{d}{dx} (uv) = v \frac{du}{dx} + u \frac{dv}{dx}
\]

\[
d(uv) = v \, du + u \, dv
\]

\[
\int d(uv) = \int (v \, du + u \, dv)
\]

\[
uv = \int v \, du + \int u \, dv
\]

\[
\int u \, dv = uv - \int v \, du
\]
Integration by Parts Formula

**Theorem**

If \( u = f(x) \) and \( v = g(x) \) and \( f' \) and \( g' \) are continuous then

\[
\int u \, dv = uv - \int v \, du
\]

or alternatively

\[
\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx.
\]
Integration by Parts Formula

**Theorem**

If $u = f(x)$ and $v = g(x)$ and $f'$ and $g'$ are continuous then

$$\int u \, dv = uv - \int v \, du$$

or alternatively

$$\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx.$$

**Remark:** the derivative of $g(x)$ in the integral on the left-hand side has moved over to $f(x)$ in the integral on the right-hand side.
Further Remarks

- When trying to apply integration by parts we must designate part of the integrand to be $dv$ and the rest $u$.
- Usually $dv$ is the most complicated part of the integrand that we can integrate using an elementary technique.
Examples

Example

Use integration by parts to evaluate the following indefinite integrals.

1. \[ \int x \sin x \, dx \]
2. \[ \int \ln x \, dx \]
3. \[ \int \cos^{-1} x \, dx \]
Integrate by parts choosing

\[ u = x \quad v = -\cos x \]
\[ du = dx \quad dv = \sin x \, dx \]

then

\[
\int x \sin x \, dx = x (-\cos x) - \int (-\cos x) \, dx
\]
\[
= -x \cos x + \int \cos x \, dx
\]
\[
= -x \cos x + \sin x + C.
\]
Integrate by parts choosing

\[ u = \ln x \quad \quad \quad \quad v = x \]
\[ du = \frac{1}{x} \, dx \quad \quad \quad \quad dv = dx \]

then

\[ \int \ln x \, dx = x \ln x - \int x \left( \frac{1}{x} \right) \, dx \]
\[ = x \ln x - \int 1 \, dx \]
\[ = x \ln x - x + C. \]
Integrate by parts choosing

\[
\begin{align*}
  u &= \cos^{-1} x \\
  du &= \frac{-1}{\sqrt{1-x^2}} \, dx \\
  v &= x \\
  dv &= dx
\end{align*}
\]

then

\[
\begin{align*}
  \int \cos^{-1} x \, dx &= x \cos^{-1} x - \int x \left( \frac{-1}{\sqrt{1-x^2}} \right) \, dx \\
  &= x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} \, dx.
\end{align*}
\]

Now integrate by substitution letting

\[
\begin{align*}
  w &= 1 - x^2 \\
  -\frac{1}{2} \, dw &= x \, dx
\end{align*}
\]
\[
\int \cos^{-1} x \, dx = x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} \, dx
\]
\[
= x \cos^{-1} x - \frac{1}{2} \int w^{-1/2} \, dw
\]
\[
= x \cos^{-1} x - \sqrt{w} + C
\]
\[
= x \cos^{-1} x - \sqrt{1-x^2} + C
\]
Remark: Sometimes integration by parts must be used more than once.

Example
Use integration by parts to evaluate the following indefinite integrals.

1. \[ \int x^2 \cos x \, dx \]
2. \[ \int e^x \sin x \, dx \]
Tabular Integration

Consider \( \int x^2 \cos x \, dx \) and suppose we arrange our work in the following table:

<table>
<thead>
<tr>
<th>u</th>
<th>dv</th>
<th>sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td>( \cos x )</td>
<td></td>
</tr>
<tr>
<td>( \cos x )</td>
<td>dv</td>
<td></td>
</tr>
<tr>
<td>( x^2 )</td>
<td>( \sin x )</td>
<td>+</td>
</tr>
<tr>
<td>( 2x )</td>
<td>( -\cos x )</td>
<td>-</td>
</tr>
<tr>
<td>( 2 )</td>
<td>( -\sin x )</td>
<td>+</td>
</tr>
<tr>
<td>0</td>
<td>( \cos x )</td>
<td>-</td>
</tr>
</tbody>
</table>

If we multiply across the completed rows and add these products we get

\[
\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2\sin x + C.
\]

**Note:** differentiate down the first column, antidifferentiate down the second column, alternate signs (starting with +) in the third column.
Example

Use tabular integration (or repeated integration by parts) to evaluate \( \int x^3 e^x \, dx \).
Often common integration formulas are expressed recursively. For example:

\[ \int \cos^n x \, dx = \int \cos^{n-1} x \cos x \, dx \]

using integration by parts with

\[
\begin{align*}
u &= \cos^{n-1} x \\
v &= \sin x \\
du &= -(n-1) \sin x \cos^{n-2} x \, dx \\
dv &= \cos x \, dx
\end{align*}
\]

we obtain

\[ \int \cos^n x \, dx = \sin x \cos^{n-1} x + \int (n-1) \sin^2 x \cos^{n-2} x \, dx \]
\[ \int \cos^n x \, dx = \sin x \cos^{n-1} x + \int (n - 1) \sin^2 x \cos^{n-2} x \, dx \]

\[ = \sin x \cos^{n-1} x + (n - 1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx \]

\[ = \sin x \cos^{n-1} x + (n - 1) \int \cos^{n-2} x \, dx \]

\[ - (n - 1) \int \cos^n x \, dx \]

\[ n \int \cos^n x \, dx = \sin x \cos^{n-1} x + (n - 1) \int \cos^{n-2} x \, dx \]

\[ \int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n - 1}{n} \int \cos^{n-2} x \, dx \]
Integration by parts is easily adapted to definite integrals:

\[ \int_{x=a}^{x=b} u \, dv = uv \bigg|_{x=a}^{x=b} - \int_{x=a}^{x=b} v \, du \]
Example

Use integration by parts to evaluate the following definite integrals.

1. \( \int_{0}^{\pi/2} x \sin x \, dx \)
2. \( \int_{-1}^{0} 3xe^{2x} \, dx \)
3. \( \int_{1}^{e} \sqrt{x} \ln x \, dx \)
Homework

- Read Section 6.2
- Exercises 1–57 odd.