1. (6 points) Find the area of the region inside of one loop of the polar coordinate equation \( r^2 = 4 \cos 2\theta \).

\[
\begin{align*}
A &= \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta \\
&= \frac{1}{2} \int_{-\pi/4}^{\pi/4} 4\cos 2\theta \, d\theta \\
&= 2 \int_{-\pi/4}^{\pi/4} \cos 2\theta \, d\theta \\
&= \sin 2\theta \bigg|_{-\pi/4}^{\pi/4} \\
&= \sin \frac{\pi}{2} - \sin \left(-\frac{\pi}{2}\right) \\
A &= 2
\end{align*}
\]
2. (6 points each) Determine if the following infinite series are divergent, conditionally convergent, or absolutely convergent. You must justify your answers.

(a) \( \sum_{k=2}^{\infty} (-1)^k \frac{\sqrt[k]{k-1}}{k^2 - 1} \)

Let \( a_k = \frac{\sqrt[k]{k-1}}{k^2 - 1} \) and \( b_k = \frac{1}{k^{5/3}} \)

\[ \lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{k^{5/3} \sqrt[k]{k-1}}{k^2 - 1} = 1 > 0 \]

By the Limit Comparison Test, both series converge. Thus the original series converges absolutely.

(b) \( \sum_{k=1}^{\infty} \frac{3^{k-1}}{k^2 + 9} \)

Let \( a_k = \frac{3^{k-1}}{k^2 + 9} \).

\[ \lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim_{k \to \infty} \frac{3 (k^2 + 9)}{(k+1)^2 + 9} = 3 > 1. \]

This series diverges by the Ratio Test.
(c) \[ \sum_{k=1}^{\infty} \frac{(-1)^{k-1} k}{k^2 + 1} \] Let \( a_k = \frac{\frac{\sqrt{k}}{k^2 + 1}}{b_k = \frac{1}{k}}. \]

Since \( \lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\frac{\sqrt{k}}{k^2 + 1}}{\frac{1}{k}} = 1 > 0, \)
the series \( \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2 + 1} \) diverges.

Note that \( a_k = a_{k+1} \) for \( k \geq 1 \) and \( \lim_{k \to \infty} a_k = 0. \)
By the Alternating Series Test the original series converges conditionally.

(d) \[ \sum_{k=1}^{\infty} k^{-2} e^{1/k} \] Let \( f(x) = \frac{e^{1/x}}{x^2} > 0 \) for \( x \geq 1. \)

\[ \int_{1}^{\infty} \frac{e^{1/x}}{x^2} \, dx = \lim_{R \to \infty} \int_{1}^{R} \frac{e^{1/x}}{x^2} \, dx = \lim_{R \to \infty} -e^{1/x} \bigg|_{1}^{R} \]
\[ = \lim_{R \to \infty} (-e^{-1} + e) = e-1 < \infty. \]
By the Integral Test the series converges absolutely.
3. (5 points) An aquarium has a rectangular base of width 2 feet and length 4 feet. The sides are rectangular and have height 3 feet. The aquarium is filled to a depth of 2.5 feet with water weighing 62.5 pounds per cubic foot. Find the amount of work required to pump all the water over the top of the aquarium.

Weight of water: \(62.5 \times (2)(4)(2.5) = 1250 \text{ lbs.}\)

Center of mass at \(y = 1.25 \text{ ft.}\)

Work = \((3-1.25)\times1250) = 2187.5 \text{ ft-lbs.}\)
4. (5 points each) Evaluate the following indefinite integrals.

(a) \[ \int \sin^3 x \cos^3 x \, dx = \int \sin^3 x (1-\sin^2 x) \cos x \, dx \]
\[ = \int u^3 (1-u^2) \, du \]
\[ = \int u^3 - u^5 \, du \]
\[ = \frac{1}{4} u^4 - \frac{1}{6} u^6 + C \]
\[ = \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C \]

Alternatively
\[ = \int (1-\cos^2 x) \cos^3 x \sin x \, dx \quad u = \cos x \]
\[ = \int (1-u^2) u^3 \, du \]
\[ = -\frac{1}{4} u^4 + \frac{1}{6} u^6 + C = -\frac{1}{4} \cos^4 x + \frac{1}{6} \cos^6 x + C \]

(b) \[ \int \frac{x}{25-9x^2} \, dx \]
\[ u = 25-9x^2 \]
\[ -\frac{1}{18} \, du = x \, dx \]
\[ = -\frac{1}{18} \int \frac{1}{u} \, du \]
\[ = -\frac{1}{18} \ln |u| + C \]
\[ = -\frac{1}{18} \ln |25-9x^2| + C \]
(c) \[ \int \frac{1}{\sqrt{25 - 9x^2}} \, dx \]

\[ = \left( \frac{\frac{5}{3} \cos \theta}{\sqrt{25 - 9\left(\frac{5}{3} \sin \theta\right)^2}} \right) \, d\theta \]

\[ = \frac{\frac{5}{3}}{\frac{5}{3}} \cdot \frac{1}{5} \int \frac{\cos \theta}{\sqrt{1 - \sin^2 \theta}} \, d\theta \]

\[ = \frac{1}{3} \int 1 \, d\theta \]

\[ = \frac{\theta}{3} + C = \frac{1}{3} \sin^{-1} \left( \frac{3x}{5} \right) + C \]

5. (5 points) The region bounded by the graph of \( f(x) = \sqrt{1 + \cos 2x} \), the x-axis, and the lines \( x = 0 \) and \( x = \pi/2 \) is revolved around the x-axis. Find the exact volume of the resulting solid of revolution.

\[ V = \pi \int_0^{\pi/2} \left( \sqrt{1 + \cos 2x} \right)^2 \, dx \]  

\[ = \pi \int_0^{\pi/2} (1 + \cos 2x) \, dx \]

\[ = \pi \left( \frac{\pi}{2} \right) \]

\[ = \frac{\pi^2}{2} \]
6. (6 points each) Find the values of the following limits, if they exist.

(a) \( \lim_{x \to 0} \frac{\sin x - x}{\tan x - x} \)

\[= \lim_{x \to 0} \frac{\cos x - 1}{\sec^2 x - 1} \quad \text{indeterminate} \]

\[= \lim_{x \to 0} \frac{-\sin x}{2\sec^2 x \tan x} \]

\[= \lim_{x \to 0} \frac{-1}{2} \cos^3 x \]

\[= -\frac{1}{2} \]

(b) \( \lim_{x \to 0} x \csc x \)

\[= \lim_{x \to 0} \frac{x}{\sin x} \quad \text{indeterminate} \]

\[= \lim_{x \to 0} \frac{1}{\cos x} \]

\[= 1 \]
(c) \[ \lim_{x \to 1^-} (1 - x)^{\ln x} = \text{indeterminate} \]

Let \( y = (1 - x)^{\ln x} \)

\[ \ln y = (\ln x) \ln (1 - x) \]

\[ \lim_{x \to 1^-} (\ln x) \ln (1 - x) = \text{indeterminate} \]

\[ = \lim_{x \to 1^-} \frac{\ln (1 - x)}{(\ln x)^{-1}} = \text{indeterminate} \]

\[ = \lim_{x \to 1^-} \frac{-1}{1 - x} \cdot \frac{-1}{x (\ln x)^2} = \text{indeterminate} \]

\[ = \lim_{x \to 1^-} \frac{x (\ln x)^2}{1 - x} = \text{indeterminate} \]

\[ = \lim_{x \to 1^-} \frac{(\ln x)^2 + 2 \ln x}{-1} = 0 \]

Therefore \( \lim_{x \to 1^-} (1 - x)^{\ln x} = e^0 = 1 \).
7. (5 points each) The position of a moving point at time $t$ is given by

\[
\begin{cases}
  x = 2 \sin t \\
  y = \sin^2 t.
\end{cases}
\]

(a) Find the speed of the point at $t = \pi/3$.

\[
\begin{align*}
\frac{dx}{dt} &= 2 \cos t \\
\frac{dy}{dt} &= 2 \sin t \cos t \\
\text{Speed} &= \sqrt{(2 \cos \frac{\pi}{3})^2 + (2 \sin \frac{\pi}{3} \cos \frac{\pi}{3})^2} \\
&= \sqrt{1 + (\sqrt{3})^2} \\
&= \frac{\sqrt{7}}{2}
\end{align*}
\]

(b) Find the distance the point travels for $t$ in the interval $[0, \pi/2]$. You may evaluate this result numerically.

\[
\begin{align*}
S &= \int_{0}^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \\
&= \int_{0}^{\pi/2} \sqrt{(2 \cos t)^2 + (2 \sin t \cos t)^2} \, dt \\
&= 2 \int_{0}^{\pi/2} \cos t \sqrt{1 + \sin^2 t} \, dt \\
&\approx 2.2956
\end{align*}
\]

(c) Find the slope of the tangent line to the path of the point at $t = \pi/6$.

\[
m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2 \sin t \cos t}{2 \cos t} = \sin t
\]

When $t = \pi/6$, $m = \frac{1}{2}$.\n
If you are a MATH major, please print your name here:__________________________

8. (6 points) Use a partial fractions decomposition to compute the following indefinite integral. Simplify your answer.

\[
\int \frac{2x + 1}{x(x^2 + 1)} \, dx
\]

\[
\frac{2x + 1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}
\]

\[
2x + 1 = A(x^2 + 1) + (Bx + C)x
\]

\[A_{x=0}: \quad 1 = A \implies 2x + 1 = x^2 + 1 + (Bx + C)x
\]

\[2 = (1+B)x + C
\]

Thus \(B = -1\) and \(C = 2\).

\[
\int \frac{2x + 1}{x(x^2 + 1)} \, dx = \int \frac{1}{x} + \frac{-x + 2}{x^2 + 1} \, dx
\]

\[
= \left( \ln |x| \right) - \left( \frac{1}{2} \ln (x^2 + 1) \right) + 2 \tan^{-1} x + C
\]
9. (6 points) Use the ratio test to determine the radius of convergence for the following power series. Also determine the interval of convergence.

\[
\lim_{k \to \infty} \left| \frac{(-1)^{k+1}(x-3)^{k+1}}{(k+1)4^{k+1}} \right| = \lim_{k \to \infty} \frac{1}{4} |x-3| = \frac{1}{4} |x-3|
\]

The power series converges absolutely if \( \frac{1}{4} |x-3| < 1 \)

\[|x-3| < 4\]

where the radius of convergence, \( r = 4 \).

Let \( x = -1 \), \( \sum_{k=1}^{\infty} \frac{(-1)^k}{k4^k} (1-3)^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k4^k} \)

= \sum_{k=1}^{\infty} \frac{1}{k} \quad \text{diverges.}

Let \( x = 7 \), \( \sum_{k=1}^{\infty} \frac{(-1)^k}{k4^k} (7-3)^k = \sum_{k=1}^{\infty} \frac{(-1)^k4^k}{k4^k} \)

= \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \quad \text{converges.}

Interval of convergence: \( -1 < x \leq 7 \).