Please answer the following questions. Your answers will be evaluated on their correctness, completeness, and use of mathematical concepts we have covered. Please show all work and write out your work neatly. Answers without supporting work will receive no credit. The point values of the problems are listed in parentheses.

1. (9 points) Use a known Taylor series to find the value of the following limit.

\[
\lim_{x \to 0} \frac{\cos x^2 - 1}{x^4} = \lim_{x \to 0} \left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots \right) - 1
\]

\[= \lim_{x \to 0} \frac{-\frac{x^4}{2} + \frac{x^8}{24} - \cdots}{x^4} = \lim_{x \to 0} \left(-\frac{1}{2} + \frac{x^4}{24} - \cdots \right) = -\frac{1}{2}
\]

2. (9 points) Use a known Taylor series to find the Taylor series with \( c = 0 \) for the function \( f(x) = xe^{-x^2} \).

\[
\text{Note: } e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \Rightarrow \quad e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}
\]

\[
\text{Thus } f(x) = xe^{-x^2} = x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!}
\]
3. (10 points) Find the Maclaurin series for \( \cosh x \).

Let \( f(x) = \cosh x \).

\[
 f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k
\]

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<tr>
<th>( k )</th>
<th>( f^{(k)}(0) )</th>
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<td>1</td>
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<td>4</td>
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Thus \( \cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \).

Alternatively, by definition \( \cosh x = \frac{1}{2} (e^x + e^{-x}) \) and thus

\[
 \cosh x = \frac{1}{2} \sum_{k=0}^{\infty} \frac{x^k}{k!} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}
\]

\[
 = \frac{1}{2} \sum_{k=0}^{\infty} \frac{x^k + (-1)^k x^k}{k!}
\]

\[
 = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(1 + (-1)^k) x^k}{k!}
\]

Note that \( 1 + (-1)^k = \begin{cases} 2, & k \text{ even} \\ 0, & k \text{ odd} \end{cases} \)

\[
 = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}
\]
4. (9 points) Determine the interval of convergence and the function to which the following series converges.

\[ \sum_{k=0}^{\infty} (2x - 1)^k \]

Treated as a geometric series with \( a = 1 \) and \( r = 2x - 1 \),

\[ \sum_{k=0}^{\infty} (2x - 1)^k = \frac{1}{1-(2x-1)} = \frac{1}{2-2x} \text{ if } |2x-1| < 1. \]

\(-1 < 2x - 1 < 1\)
\(0 < 2x < 2\)
\(0 < x < 1\) is the interval of convergence.

5. (9 points) Find a power series representation for the function \( f(x) = \frac{2x}{1-x^3} \). State the radius of convergence of the power series.

Treated as the sum of a geometric series with \( a = 2x \) and \( r = x^3 \),

\[ f(x) = \frac{2x}{1-x^3} = \sum_{k=0}^{\infty} (2x)(x^3)^k = \sum_{k=0}^{\infty} 2x \cdot 3^{k+1}. \]

The series converges when \( |r| = |x^3| < 1 \)
\( \Rightarrow 1|x| < 1 \). The radius of convergence is 1.
6. (9 points) Determine the interval of convergence of the following power series.

\[
\lim_{k \to \infty} \left| \frac{(-1)^k (x-1)^{k+1}}{(k+1)3^k} \right| = \lim_{k \to \infty} \frac{k}{(k+1)} |x-1|
\]

\[
= \frac{1}{3} |x-1| < 1 \Rightarrow |x-1| < 3 \Rightarrow -3 < x < 4.
\]

\(\gamma x = -2, \sum_{k=1}^{\infty} \frac{(-1)^k}{k3^k} = -\sum_{k=1}^{\infty} \frac{1}{k}, \text{ Diverges (Harmonic Series)}\)

\(\gamma x = 4, \sum_{k=1}^{\infty} \frac{(-1)^k}{k3^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}, \text{ Converges (Alt. Harmonic Series)}\)

Interval of convergence: \(-2 < x \leq 4\).

7. (9 points) Find a power series in \(x\) representation of \(f(x) = \ln(1 + x^2)\).

\[f'(x) = \frac{2x}{1+x^2} = \frac{2x}{1-(-x^2)} = \sum_{k=0}^{\infty} 2x(-x^2)^k = \sum_{k=0}^{\infty} 2(-1)^k x^{2k+1}, \text{ if } |x| < 1.\]

Since \(\gamma x =\)

\[\sum_{k=0}^{\infty} \frac{2(-1)^k}{2k+2} x^{2k+2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{2k+2} \]

\[\text{if } |x| < 1.\]
8. (9 points each) Determine whether the following infinite series are absolutely convergent, conditionally convergent, or divergent. You must justify your answers by naming the test(s) for convergence or divergence you use.

(a) \( \sum_{k=1}^{\infty} (-1)^k \frac{3^k}{k!} \)

Ratio test: \[ \lim_{k \to \infty} \left| \frac{\frac{(-1)^{k+1} 3^{k+1}}{(k+1)!}}{\frac{(-1)^k 3^k}{k!}} \right| = \lim_{k \to \infty} \left| \frac{k!}{(k+1)!} \cdot \frac{3}{3^k} \right| \\
= \lim_{k \to \infty} \frac{3}{k+1} = 0 < 1 \]

Series converges absolutely.

(b) \( \sum_{k=1}^{\infty} \frac{\sin k}{k^2} \)

Consider \( \sum_{k=1}^{\infty} \left| \frac{\sin k}{k^2} \right| \). Since \( \left| \frac{\sin k}{k^2} \right| \leq \frac{1}{k^2} \)

and \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) converges by the \( p \)-series test, then \( \sum_{k=1}^{\infty} \frac{\sin k}{k^2} \) converges absolutely by the Comparison Test.
(c) \[ \sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k} \]

Consider \[ \sum_{k=2}^{\infty} \left| \frac{(-1)^k}{\ln k} \right| = \sum_{k=2}^{\infty} \frac{1}{\ln k} \]. Since \[ \frac{1}{\ln k} > \frac{1}{k} \]

for all \( k \) and \( \sum_{k=2}^{\infty} \frac{1}{k} \) diverges (Harmonic series), the original series cannot converge absolutely. So it is an alternating series. \[ \frac{1}{\ln (k+1)} \leq \frac{1}{\ln k} \] for all \( k \) and \[ \lim_{k \to \infty} \frac{1}{\ln k} = 0 \]. Thus by the Alternating Series Test, the original series converges conditionally.

9. (9 points) Determine the minimum number of terms necessary to estimate the sum of the following series to within \( 10^{-4} \). Then using the partial sum with the minimum number of terms, estimate the sum of the series.

\[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 4}{k^4} \]

This is a convergent alternating series, thus

\[ |S - S_n| \leq a_{n+1} = \frac{4}{(n+1)^4} < 10^{-4} \]

\[ \frac{(n+1)^4}{4} > 10^4 \]

\[ (n+1)^4 > 40000 \]

\[ n > 13.14 \implies n \geq 14. \]

\[ S_{14} = \sum_{k=1}^{14} \frac{(-1)^{k+1} 4}{k^4} = 3.7881 \]