Please answer the following questions. Your answers will be evaluated on their correctness, completeness, and use of mathematical concepts we have covered. Please show all work and write out your work neatly. Answers without supporting work will receive no credit. The point values of the problems are listed in parentheses.

1. (12 points) Find parametric equations in Cartesian coordinates for the portion of the parabola \( y = x^2 + 3 \) between (1, 4) and (2, 7).

\[
\begin{cases}
  x = t \\
  y = t^2 + 3
\end{cases} \quad \text{for} \quad 1 \leq t \leq 2
\]

2. (12 points) Find the arc length of the curve described parametrically by

\[
\begin{cases}
  x = t^2 \cos t \\
  y = t^2 \sin t
\end{cases} \quad \text{for} \quad -1 \leq t \leq 1.
\]

You may approximate the value on your calculator.

\[
S = \int_{-1}^{1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

\[
= \int_{-1}^{1} \sqrt{\left(2t - t^2 \sin t\right)^2 + \left(2t \cos t + t^2 \sin t\right)^2} \, dt
\]

\[
= \int_{-1}^{1} \sqrt{4t^2 \cos^2 t + 4t^2 \sin^2 t + t^4 \cos^2 t + t^4 \sin^2 t} \, dt
\]

\[
= \int_{-1}^{1} \sqrt{t^4 + 4t^2} \, dt
\]

\[
= 2 \int_{0}^{1} t \sqrt{t^2 + 4} \, dt = \frac{2}{3} (t^2 + 4)^{3/2} \bigg|_{0}^{1} = \frac{2}{3} (5\sqrt{5} - 8)
\]

\[
\approx 2.1202
\]
3. (7 points each) Consider the parametric equations

\[ \begin{align*}
  x &= t^2 - 1 \\
  y &= t^4 - 4t^2 
\end{align*} \]

(a) Identify all the points at which the curve has a horizontal tangent, if any.

\[ \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{4t^3 - 8t}{2t} \]

Suppose \( \frac{dy}{dt} = 4t^3 - 8t = 0 = 4t(t^2 - 2) \Rightarrow t=0 \text{ or } t = \pm \sqrt{2}. \)

\[ \lim_{{t \to 0}} \frac{4t^3 - 8t}{2t} = \lim_{{t \to 0}} \left(2t^2 - 4\right) = -4 \neq 0 \]

Horizontal tangents occur when \( t = \pm \sqrt{2}. \)

Since \((x, y) = (1, 1-8) = (1, -7).\)

(b) Identify all the points at which the curve has a vertical tangent, if any.

\[ \frac{dx}{dt} = 0 \text{ only when } t=0; \text{ however, } \]

\[ \lim_{{t \to 0}} \frac{dx}{dt} = -4 \neq \pm \infty. \text{ Hence there are no vertical tangents.} \]
4. (12 points) Find a polar coordinate equation for the curve described in Cartesian coordinates as $x^2 + y^2 = x + y$.

Recall that $x^2 + y^2 = r^2$ and $x = r \cos \theta$ and $y = r \sin \theta$.

Hence

$$\begin{align*}
x^2 + y^2 &= x + y \\
r^2 &= r \cos \theta + r \sin \theta.
\end{align*}$$

If $r \neq 0$, then $r = \cos \theta + \sin \theta$.

5. (12 points) Find the slope of the tangent line to the polar coordinate curve $r = 2 - 2 \cos \theta$ at $\theta = \pi/6$.

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}$$

$$= \frac{(2 \sin \theta) \sin \theta + (2 - 2 \cos \theta) \cos \theta}{(2 \sin \theta) \cos \theta - (2 - 2 \cos \theta) \sin \theta}$$

When $\theta = \pi/6$, $\frac{dy}{dx} = \frac{(2 \sin \pi/6) \sin \pi/6 + (2 - 2 \cos \pi/6) \cos \pi/6}{(2 \sin \pi/6) \cos \pi/6 - (2 - 2 \cos \pi/6) \sin \pi/6}$

$$= \frac{(1)(1/2) + (2 - \sqrt{3})(\sqrt{3}/2)}{(1)(\sqrt{3}/2) - (2 - \sqrt{3})(1/2)}$$

$$= \frac{\frac{1}{2} + (2 - \sqrt{3})(\sqrt{3}/2)}{\frac{\sqrt{3}}{2} - (2 - \sqrt{3})(1/2)} \cdot \frac{2}{2}$$

$$= \frac{1 + (2 - \sqrt{3})\sqrt{3}}{\frac{\sqrt{3}}{2} - (2 - \sqrt{3})(1/2)} = \frac{2\sqrt{3} - 2}{2\sqrt{3} - 2}$$

$$= 1$$
6. (14 points) Find the exact area inside of the curve \( r = 2 \sin 2\theta \) and outside of \( r = 1 \).

The curves intersect when

\[
\begin{align*}
1 &= 2 \sin 2\theta \\
\frac{1}{2} &= \sin 2\theta \\
2\theta &= \frac{\pi}{6} \quad \text{or} \quad 2\theta = \frac{5\pi}{6} \\
\theta &= \frac{\pi}{12} \quad \text{or} \quad \theta = \frac{5\pi}{12}
\end{align*}
\]

\[
A = 4 \left[ \frac{1}{2} \left( \int_{\pi/12}^{5\pi/12} (2\sin 2\theta)^2 - 1 \right) \, d\theta \right]
\]

\[
= 2 \left( \int_{\pi/12}^{5\pi/12} (4\sin^2 2\theta - 1) \, d\theta \right)
\]

\[
= 8 \left( \int_{\pi/12}^{5\pi/12} \sin^2 2\theta \, d\theta \right) - 2 \left( \int_{\pi/12}^{5\pi/12} 1 \, d\theta \right)
\]

\[
= 4 \left( \int_{\pi/12}^{5\pi/12} (1 - \cos 4\theta) \, d\theta \right) - 2 \left( \int_{\pi/12}^{5\pi/12} 1 \, d\theta \right)
\]

\[
= 2 \left( \int_{\pi/12}^{5\pi/12} 1 \, d\theta \right) - 4 \left( \int_{\pi/12}^{5\pi/12} \cos 4\theta \, d\theta \right)
\]

\[
= 2 \left( \frac{5\pi}{12} - \frac{\pi}{12} \right) - \sin 4\theta \bigg|_{\pi/12}^{5\pi/12}
\]

\[
= 2 \left( \frac{4\pi}{12} \right) - \left( \sin \frac{5\pi}{12} - \sin \frac{\pi}{12} \right)
\]

\[
= \frac{2\pi}{3} - \left[ \frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right]
\]

\[
A = \sqrt{3} + \frac{2\pi}{3} \approx 3.82645
\]
7. (12 points) Find each focus and vertex of the conic section whose equation is

$$\frac{(x+3)^2}{9} - \frac{(y-2)^2}{4} = 1.$$ 

The equation can be written in standard form:

$$\frac{(x-(-3))^2}{3^2} - \frac{(y-2)^2}{2^2} = 1 \quad \text{(Hyperbola)}$$

Let \((x_0, y_0) = (-3, 2)\) and \(c = \sqrt{9+4} = \sqrt{13}\)

foci: \((-3 \pm \sqrt{13}, 2)\)

vertex: \((-3 \pm 3, 2) \Rightarrow (-6, 2)\) and \((0, 2)\).

8. (12 points) Find the equation of the curve in Cartesian coordinates describing all the points in the plane which are equidistant from the point \((1, 0)\) and the line \(y = -2\).

\[
\sqrt{(x-1)^2 + y^2} = y - (-2)
\]

\[
(x-1)^2 + y^2 = (y+2)^2
\]

\[
= y^2 + 4y + 4
\]

\[
(x-1)^2 = 4y + 4
\]

\[
\frac{1}{4} (x-1)^2 - 1 = y
\]

(Parabola)