1. (8 points) Use integration by parts to compute the following indefinite integral.

$$\int (2x^2 - 9)e^{2x} \, dx$$

Using tabular integration

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u$</td>
<td>$dv$</td>
</tr>
<tr>
<td>---</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td></td>
<td>$2x^2 - 9$</td>
<td>$\frac{1}{4}e^{2x}$</td>
</tr>
<tr>
<td></td>
<td>$4x$</td>
<td>$\frac{1}{2}e^{2x}$</td>
</tr>
<tr>
<td></td>
<td>$4$</td>
<td>$\frac{1}{8}e^{2x}$</td>
</tr>
</tbody>
</table>

we see that

$$\int (2x^2 - 9)e^{2x} \, dx = \frac{2x^2 - 9}{2}e^{2x} - xe^{2x} + \frac{1}{2}e^{2x} + C$$

$$= \left(x^2 - x - 4\right)e^{2x} + C$$
2. (8 points) Use a partial fractions decomposition to compute the following indefinite integral. Simplify your answer.

\[ \int \frac{x^2 - 4}{x^3 - 2x^2 - 15x} \, dx \]

Note that

\[ \frac{x^2 - 4}{x^3 - 2x^2 - 15x} = \frac{x^2 - 4}{x(x + 3)(x - 5)} = \frac{A}{x} + \frac{B}{x + 3} + \frac{C}{x - 5} \]

\[ x^2 - 4 = A(x + 3)(x - 5) + Bx(x - 5) + Cx(x + 3). \]

If we let \( x = 0 \) then

\[ -4 = -15A \]
\[ A = \frac{4}{15}. \]

If we let \( x = -3 \) then

\[ 5 = 24B \]
\[ B = \frac{5}{24}. \]

If we let \( x = 5 \) then

\[ 21 = 40C \]
\[ C = \frac{21}{40}. \]

Therefore we have

\[ \int \frac{x^2 - 4}{x^3 - 2x^2 - 15x} \, dx = \int \left( \frac{4/15}{x} + \frac{5/24}{x + 3} + \frac{21/40}{x - 5} \right) \, dx \]
\[ = \frac{4}{15} \ln |x| + \frac{5}{24} \ln |x + 3| + \frac{21}{40} \ln |x - 5| + C. \]
3. (8 points) The region bounded between the graphs of \( x = 4 - y^2 \) and \( x = y^2 - 4 \) is revolved around the \( x \)-axis. Find the exact volume of the solid of revolution.

\[
V = 2 \left[ 2\pi \int_{0}^{2} y(4 - y^2) \, dy \right] \\
= 4\pi \int_{0}^{2} 4y - y^3 \, dy \\
= \left( 4\pi \left[ 2y^2 - \frac{1}{4}y^4 \right] \right)_{0}^{2} \\
= 4\pi (8 - 4) \\
= 16\pi
\]
4. (6 points) Find the arclength of the portion of the graph of \( y = x^2 + x \) where \(-1 \leq x \leq 1\). You may numerically approximate the arc length after setting the appropriate definite integral.

\[
\begin{align*}
\ s \ &= \int_{-1}^{1} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \ dx \\
&= \int_{-1}^{1} \sqrt{1 + (2x + 1)^2} \ dx \\
&\approx 3.40022
\end{align*}
\]

5. (8 points) Determine whether the following series converges absolutely, converges conditionally, or diverges.

\[
\sum_{k=1}^{\infty} \frac{(-1)^k k}{3^k}
\]

Applying the Ratio Test for absolute convergence we have

\[
\lim_{k \to \infty} \left| \frac{\frac{(-1)^{k+1} (k+1)}{3^{k+1}}}{\frac{(-1)^k k}{3^k}} \right| = \lim_{k \to \infty} \left| \frac{(-1)^{k+1} (k+1)3^k}{(-1)^k 3^{k+1} k} \right| = \lim_{k \to \infty} \frac{1}{3} \left( 1 + \frac{1}{k} \right) = \frac{1}{3} < 1.
\]

Thus the infinite series converges absolutely.
6. (8 points) Find a power series in $x$ representation for the function

$$f(x) = \frac{2}{1 + 4x^2}.$$ 

State the radius of convergence for the power series.

$$f(x) = \frac{2}{1 + 4x^2} = \frac{2}{1 - (-4x^2)} = \sum_{k=0}^{\infty} 2(-4x^2)^k \quad \text{(if } | - 4x^2 | < 1 \text{)}$$

$$= \sum_{k=0}^{\infty} (-1)^k 2(4)^k x^{2k}$$

$$= \sum_{k=0}^{\infty} (-1)^k 2^{2k+1} x^{2k}$$

The series converges absolutely when $| - 4x^2 | < 1$ which implies the radius of convergence is $r = 1/2$. 

7. (6 points) Consider the curve described by the parametric equations

\[ x \, = \, t^3 - 3t \]
\[ y \, = \, t^2 + 2t \]

Find the slope of the tangent line to the graph of the parametric curve when \( t = 1 \).

The slope of the tangent line is undefined since

\[
\lim_{t \to 1} \frac{dy}{dx} = \frac{\lim_{t \to 1} \frac{dy}{dt}}{\lim_{t \to 1} \frac{dx}{dt}}
\]
\[
= \lim_{t \to 1} \frac{2t + 2}{3t^2 - 3}
\]
\[
= -\infty.
\]
8. (8 points) Identify the type of conic section represented by the equation below. Find all the foci, vertices, asymptotes, and the directrix if appropriate.

\[
\frac{x^2}{9} - \frac{(y - 2)^2}{4} = 1
\]

The conic section is a hyperbola.

The foci are located at

\[
(x_0 \pm c, y_0) = (0 \pm \sqrt{9 + 4}, 2) = (\pm \sqrt{13}, 2).
\]

The vertices are located at

\[
(x_0 \pm a, y_0) = (0 \pm 3, 2) = (\pm 3, 2).
\]

The asymptotes are

\[
y = \pm \frac{b}{a}(x - x_0) + y_0 = \pm \frac{2}{3}x + 2.
\]
9. (8 points) A water tower is spherical in shape with a radius of 50 feet, extending from 200 feet to 300 feet above ground. Compute the work done in filling the bottom half of the tank with water from the ground level. You may use the fact that water weighs 62.4 lbs/ft$^3$.

An equation representing the boundary of the water tank is

\[ x^2 + (y - 250)^2 = 50^2. \]

The work done filling the bottom half of the tank is

\[
W = 62.4 \int_{200}^{250} \pi (2500 - (y - 250)^2) y \, dy \\
= 62.4\pi \int_{200}^{250} (2500 - (y - 250)^2) y \, dy \\
= 62.4\pi \left( \frac{57812500}{3} \right) \\
= 3.77777 \times 10^9 \text{ ft-lbs.}
\]
10. (8 points) Evaluate the following definite integral.

\[
\int_1^\infty x e^{-3x} \, dx
\]

\[
\int_1^\infty x e^{-3x} \, dx = \lim_{R \to \infty} \int_1^R x e^{-3x} \, dx
\]

\[
= \lim_{R \to \infty} \left[ \left( -e^{-3x} \left[ \frac{x}{3} + \frac{1}{9} \right] \right) \right]_1^R
\]

\[
= \lim_{R \to \infty} \left[ \left( \frac{1}{3} + \frac{1}{9} \right) e^{-3} - e^{-3R} \left( \frac{R}{3} + \frac{1}{9} \right) \right]
\]

\[
= \frac{4}{9e^3} - \lim_{R \to \infty} e^{-3R} \left( \frac{R}{3} + \frac{1}{9} \right)
\]

\[
= \frac{4}{9e^3} - \lim_{R \to \infty} \frac{3R + 1}{9e^{3R}}
\]

\[
= \frac{4}{9e^3} - \lim_{R \to \infty} \frac{3}{27e^{3R}}
\]

\[
= \frac{4}{9e^3}
\]

\[
\approx 0.0221276
\]
11. (8 points) For the function $f(x) = 1/\sqrt{x}$, find the fourth Taylor polynomial with $c = 1$.

By definition $P_4(x)$ is

$$P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(1)}{k!} \cdot (x - 1)^k.$$ 


<table>
<thead>
<tr>
<th>$k$</th>
<th>$f^{(k)}(x)$</th>
<th>$f^{(k)}(1)$</th>
<th>$f^{(k)}(1)/k!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$x^{-1/2}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$(-1/2)x^{-3/2}$</td>
<td>$-1/2$</td>
<td>$-1/2$</td>
</tr>
<tr>
<td>2</td>
<td>$(3/4)x^{-5/2}$</td>
<td>3/4</td>
<td>3/8</td>
</tr>
<tr>
<td>3</td>
<td>$(-15/8)x^{-7/2}$</td>
<td>$-15/8$</td>
<td>$-5/16$</td>
</tr>
<tr>
<td>4</td>
<td>$(105/16)x^{-9/2}$</td>
<td>$105/16$</td>
<td>$35/128$</td>
</tr>
</tbody>
</table>

Therefore we have

$$P_4(x) = 1 - \frac{1}{2}(x - 1) + \frac{3}{8}(x - 1)^2 - \frac{5}{16}(x - 1)^3 + \frac{35}{128}(x - 1)^4.$$
12. (8 points) Evaluate the following indefinite integral.

\[ \int \cos^4 x \sin^3 x \, dx \]

\[ \int \cos^4 x \sin^3 x \, dx = \int \cos^4 x \sin^2 x \sin x \, dx \]
\[ = \int \cos^4 x (1 - \cos^2 x) \sin x \, dx \]
\[ = -\int u^4(1 - u^2) \, du \]
\[ = \int u^4(u^2 - 1) \, du \]
\[ = \int (u^6 - u^4) \, du \]
\[ = \frac{1}{7} u^7 - \frac{1}{5} u^5 + C \]
\[ = \frac{1}{7} \cos^7 x - \frac{1}{5} \cos^5 x + C \]
13. (8 points) Find the area enclosed by one leaf of the polar coordinate curve $r = \cos 2\theta$.

$$A = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta$$

$$= \frac{1}{4} \int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta) \, d\theta$$

$$= \frac{1}{4} \left( \theta + \frac{1}{4} \sin 4\theta \right) \Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{4} \left( \frac{\pi}{4} - \frac{-\pi}{4} \right)$$

$$= \frac{\pi}{8}$$

$$\approx 0.392699$$