1. (8 points) Use integration by parts to compute the following indefinite integral.

\[ \int (5 - x^2)e^{3x} \, dx \]

Using tabular integration

<table>
<thead>
<tr>
<th>( u )</th>
<th>( dv )</th>
<th>sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 5 - x^2 )</td>
<td>( e^{3x} )</td>
<td>+</td>
</tr>
<tr>
<td>( -2x )</td>
<td>( \frac{1}{3}e^{3x} )</td>
<td>-</td>
</tr>
<tr>
<td>( -2 )</td>
<td>( \frac{1}{9}e^{3x} )</td>
<td>+</td>
</tr>
</tbody>
</table>

we see that

\[ \int (5 - x^2)e^{3x} \, dx = \frac{5 - x^2}{3}e^{3x} - \frac{-2x}{9}e^{3x} + \frac{-2}{27}e^{3x} + C \]

\[ = \left( -\frac{1}{3}x^2 + \frac{2}{9}x + \frac{43}{27} \right)e^{3x} + C \]
2. (8 points) Use a partial fractions decomposition to compute the following indefinite integral. Simplify your answer.

\[
\int \frac{x^2 - 3}{x^3 + x^2 - 6x} \, dx
\]

Note that

\[
\frac{x^2 - 3}{x^3 + x^2 - 6x} = \frac{x^2 - 3}{x(x + 3)(x - 2)} = \frac{A}{x} + \frac{B}{x + 3} + \frac{C}{x - 2}
\]

\[
x^2 - 3 = A(x + 3)(x - 2) + B(x - 2) + C(x + 3).
\]

If we let \(x = 0\) then

\[
-3 = -6A \\
A = \frac{1}{2}.
\]

If we let \(x = -3\) then

\[
6 = 15B \\
B = \frac{2}{5}.
\]

If we let \(x = 2\) then

\[
1 = 10C \\
C = \frac{1}{10}.
\]

Therefore we have

\[
\int \frac{x^2 - 3}{x^3 + x^2 - 6x} \, dx = \int \left( \frac{1/2}{x} + \frac{2/5}{x + 3} + \frac{1/10}{x - 2} \right) \, dx
\]

\[
= \frac{1}{2} \ln |x| + \frac{2}{5} \ln |x + 3| + \frac{1}{10} \ln |x - 2| + C.
\]
3. (8 points) Find the volume of the solid of revolution if the region bounded by \( y = \sec x \), \( y = \sqrt{2}, \ x = -\pi/4 \), and \( x = \pi/4 \) is revolved around the \( x \)-axis.

\[
V = \pi \int_{-\pi/4}^{\pi/4} \left[ \left( \sqrt{2} \right)^2 - (\sec x)^2 \right] \, dx
\]

\[
= 2\pi \int_0^{\pi/4} \left[ 2 - \sec^2 x \right] \, dx
\]

\[
= 2\pi \left( 2x - \tan x \right) \bigg|_0^{\pi/4}
\]

\[
= 2\pi \left( \frac{\pi}{4} - \tan \frac{\pi}{4} \right)
\]

\[
= 2\pi \left( \frac{\pi}{2} - 1 \right)
\]

\[
= \pi^2 - 2\pi
\]

\[
\approx 3.58642
\]
4. (6 points) Consider the parametric equations

\[ x(t) = 2 \cos t \]
\[ y(t) = 3 \sin t \]

for \( 0 \leq t \leq \pi \). If this curve is revolved around the \( x \)-axis, it creates a surface known as an ellipsoid. Find the surface area of this ellipsoid. You may numerically approximate on your calculator the value of the appropriate definite integral.

\[
S = 2\pi \int_0^\pi |y(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt
\]

\[
= 2\pi \int_0^\pi (3 \sin t) \sqrt{(-2 \sin t)^2 + (3 \cos t)^2} \; dt
\]

\[
= 6\pi \int_0^\pi \sin t \sqrt{4 \sin^2 t + 9 \cos^2 t} \; dt
\]

\[
\approx 89.0007
\]
5. (8 points) Determine whether the following series converges absolutely, converges conditionally, or diverges.

\[ \sum_{k=1}^{\infty} \frac{(-1)^k 3k^2}{k^3 + 2k^2 + 1} \]

If we consider the absolute value series

\[ \sum_{k=1}^{\infty} \frac{3k^2}{k^3 + 2k^2 + 1} \]

and let \( a_k = \frac{3k^2}{k^3 + 2k^2 + 1} \) and let \( b_k = \frac{1}{k} \) and apply the Limit Comparison Test we see that

\[
\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\frac{3k^2}{k^3 + 2k^2 + 1}}{\frac{1}{k}} \\
= \lim_{k \to \infty} \frac{3k^3}{k^3 + 2k^2 + 1} \\
= \lim_{k \to \infty} \frac{3}{1 + \frac{2}{k} + \frac{1}{k^3}} \\
= 3.
\]

Since \( \sum_{k=1}^{\infty} b_k \) diverges then \( \sum_{k=1}^{\infty} a_k \) diverges as well. Thus the original series cannot converge absolutely.

Since \( a_k \to 0 \) as \( k \to \infty \) and \( a_{k+1} \leq a_k \) then by the Alternating Series Test \( \sum_{k=1}^{\infty} (-1)^k a_k \) converges. Hence the original series converges conditionally.

6. (8 points) Find a power series in \( x \) representation for the function

\[ f(x) = \sin \frac{2}{3} x. \]

State the radius of convergence for the power series.

Recall that

\[ \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k + 1)!} \]

which implies

\[
\sin \frac{2}{3} x = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{2}{3} x)^{2k+1}}{(2k + 1)!} \\
= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{2k+1}}{3^{2k+1}(2k + 1)!}.
\]

The radius of convergence is \( r = \infty \).
7. (6 points) Consider the curve described by the polar coordinate equation

\[ r = 2 - 3 \sin \theta \]

Find the slope of the tangent line to the graph of the polar curve when \( \theta = \pi \).

\[
\frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}
\]

Thus when \( \theta = \pi \) we have

\[
\frac{dy}{dx} = \frac{(-3 \cos \pi) \sin \pi + (2 - 3 \sin \pi) \cos \pi}{(-3 \cos \pi) \cos \pi - (2 - 3 \sin \pi) \sin \pi}
\]

\[
= \frac{(3)(0) + (2)(-1)}{(3)(-1) - (2)(0)}
\]

\[
= \frac{2}{3}
\]

8. (8 points) Identify the type of conic section represented by the equation below. Find all the foci, vertices, asymptotes, and the directrix if appropriate.

\[
\frac{(x + 1)^2}{12} + \frac{(y - 2)^2}{3} = 1
\]

The conic section is an ellipse.

**foci:** \((-4, 2)\) and \((2, 2)\)

**vertices:** \((-1 - 2\sqrt{3}, 2)\) and \((-1 + 2\sqrt{3}, 2)\)
9. (8 points) A spring has a natural length of 10 inches. An 800 pound force stretches the spring to a length of 16 inches. How much work is done in stretching the spring from a length of 12 inches to a length of 14 inches?

Using Hooke’s Law for a linear spring

\[ F = kx \]
\[ 800 = k(16 - 10) \]
\[ k = \frac{400}{3} \text{ pounds per inch.} \]

Thus the work done is

\[ W = \int_{12-10}^{14-10} kx \, dx \]
\[ = \int_2^4 400 \frac{x}{3} \, dx \]
\[ = \frac{200}{3} x^2 \bigg|_2^4 \]
\[ = \frac{200}{3} \left( 4^2 - 2^2 \right) \]
\[ = 800 \text{ inch-pounds.} \]
10. (8 points) Evaluate the following definite integral.

\[ \int_0^2 \frac{x}{\sqrt{4-x^2}} \, dx \]

\[ \int_0^2 \frac{x}{\sqrt{4-x^2}} \, dx = \lim_{R \to 2^-} \int_0^R \frac{x}{\sqrt{4-x^2}} \, dx \]

\[ = \lim_{R \to 2^-} \int_4^{4-R^2} \frac{1}{\sqrt{u}} \, du \]

\[ = - \lim_{R \to 2^-} \left( \sqrt{u} \bigg|_4^{4-R^2} \right) \]

\[ = - \lim_{R \to 2^-} (\sqrt{4-R^2} - \sqrt{4}) \]

\[ = 2 \]

11. (8 points) For the function \( f(x) = \sin \frac{x}{2} \), find the fourth Taylor polynomial with \( c = \pi/2 \).

By definition \( P_4(x) \) is

\[ P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(\pi/2)}{k!} (x - \pi/2)^k. \]

<table>
<thead>
<tr>
<th>( k )</th>
<th>( f^{(k)}(x) )</th>
<th>( f^{(k)}(\pi/2) )</th>
<th>( f^{(k)}(\pi/2)/k! )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \sin \frac{x}{2} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{2} \cos \frac{x}{2} )</td>
<td>( \frac{1}{2\sqrt{2}} )</td>
<td>( \frac{1}{2\sqrt{2}} )</td>
</tr>
<tr>
<td>2</td>
<td>( -\frac{1}{4} \sin \frac{x}{2} )</td>
<td>( -\frac{1}{4\sqrt{2}} )</td>
<td>( -\frac{1}{4\sqrt{2}} )</td>
</tr>
<tr>
<td>3</td>
<td>( -\frac{1}{8} \cos \frac{x}{2} )</td>
<td>( -\frac{1}{8\sqrt{2}} )</td>
<td>( -\frac{1}{8\sqrt{2}} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1}{16} \sin \frac{x}{2} )</td>
<td>( \frac{1}{16\sqrt{2}} )</td>
<td>( \frac{1}{16\sqrt{2}} )</td>
</tr>
</tbody>
</table>

Therefore we have

\[ P_4(x) = \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} (x - \pi/2) - \frac{1}{4\sqrt{2}} (x - \pi/2)^2 - \frac{1}{48\sqrt{2}} (x - \pi/2)^3 + \frac{1}{384\sqrt{2}} (x - \pi/2)^4. \]
12. (8 points) Evaluate the following indefinite integral.

\[
\int \tan^3 \frac{x}{3} \sec \frac{x}{3} \, dx
\]

\[
\int \tan^3 \frac{x}{3} \sec \frac{x}{3} \, dx = \int \tan^2 \frac{x}{3} \sec \frac{x}{3} \tan \frac{x}{3} \, dx
\]

\[
= \int \left( \sec^2 \frac{x}{3} - 1 \right) \sec \frac{x}{3} \tan \frac{x}{3} \, dx
\]

We may integrate by substitution by letting

\[ u = \sec \frac{x}{3} \quad \text{and} \quad 3 \, du = \sec \frac{x}{3} \tan \frac{x}{3} \, dx. \]

Thus

\[
\int \tan^3 \frac{x}{3} \sec \frac{x}{3} \, dx = \int \left( \sec^2 \frac{x}{3} - 1 \right) \sec \frac{x}{3} \tan \frac{x}{3} \, dx
\]

\[
= 3 \int (u^2 - 1) \, du
\]

\[
= u^3 - 3u + C
\]

\[
= \sec^3 \frac{x}{3} - 3 \sec \frac{x}{3} + C
\]
13. (8 points) Evaluate the following limit, if it exists.

\[
\lim_{x \to \infty} (x + e^x)^{2/x}
\]

The limit is indeterminate of the form \(\infty^0\). Let

\[y = (x + e^x)^{2/x}\]
\[\ln y = \frac{2}{x} \ln (x + e^x) = \frac{2 \ln (x + e^x)}{x}\]

then

\[
\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{2 \ln (x + e^x)}{x} \quad \text{(indeterminate } \infty/\infty)\\
= \lim_{x \to \infty} \frac{2(1 + e^x)}{x + e^x} \quad \text{(indeterminate } \infty/\infty)\\
= \lim_{x \to \infty} \frac{2 + 2e^x}{x + e^x} \quad \text{(indeterminate } \infty/\infty)\\
= \lim_{x \to \infty} \frac{2e^x}{1 + e^x} \quad \text{(indeterminate } \infty/\infty)\\
= \lim_{x \to \infty} \frac{2e^x}{e^x} \quad \text{ Thus}\\
= 2
\]

Thus

\[
\lim_{x \to \infty} (x + e^x)^{2/x} = e^2.
\]