Please answer the following questions. Your answers will be evaluated on their correctness, completeness, and use of mathematical concepts we have covered. Please show all work and write out your work neatly. Answers without supporting work will receive no credit. The point values of the problems are listed in parentheses.

1. (14 points) Find the area of the region inside the graph of $r = 3 + 2 \sin \theta$ and outside the graph of $r = 4$.

The curves intersect when

$$3 + 2 \sin \theta = 4$$
$$2 \sin \theta = 1$$
$$\sin \theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6} \text{ or } \theta = \frac{5\pi}{6}.$$
Thus the area between the curves is

\[ A = \frac{1}{2} \int_{\pi/6}^{5\pi/6} (3 + 2 \sin \theta)^2 - 4^2 \, d\theta \]

\[ = \frac{1}{2} \int_{\pi/6}^{5\pi/6} (9 + 12 \sin \theta + 4 \sin^2 \theta - 16) \, d\theta \]

\[ = \frac{1}{2} \int_{\pi/6}^{5\pi/6} (-7 + 12 \sin \theta + 4 \sin^2 \theta) \, d\theta \]

\[ = \frac{1}{2} \int_{\pi/6}^{5\pi/6} (-7 + 12 \sin \theta) \, d\theta + \frac{1}{2} \int_{\pi/6}^{5\pi/6} 4 \sin^2 \theta \, d\theta \]

\[ = \left( \frac{-7}{2} \theta - 6 \cos \theta \right) \bigg|_{\pi/6}^{5\pi/6} + \int_{\pi/6}^{5\pi/6} (1 - \cos 2\theta) \, d\theta \]

\[ = -\frac{7}{2} \left( \frac{5\pi}{6} - \frac{\pi}{6} \right) - 6 \left( \cos \frac{5\pi}{6} - \cos \frac{\pi}{6} \right) + \left( \theta - \frac{1}{2} \sin 2\theta \right) \bigg|_{\pi/6}^{5\pi/6} \]

\[ = -\frac{7\pi}{3} + 6 \frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + \left( \frac{5\pi}{6} - \frac{\pi}{6} \right) - \frac{1}{2} \left( \sin \frac{5\pi}{3} - \sin \frac{\pi}{3} \right) \]

\[ = -\frac{7\pi}{3} + 6\sqrt{3} + \frac{2\pi}{3} - \frac{1}{2} \left( \frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right) \]

\[ = -\frac{5\pi}{3} + 6\sqrt{3} + \frac{\sqrt{3}}{2} \]

\[ = \frac{13\sqrt{3}}{2} - \frac{5\pi}{3} \]

\[ \approx 6.02234 \]
2. (12 points) Find the interval of convergence of the series:

\[
\sum_{k=0}^{\infty} (-1)^k \frac{4^{2k}}{\sqrt{k+1}} x^k
\]

Applying the Ratio Test:

\[
\lim_{k \to \infty} \left| \frac{(-1)^{k+1} \frac{4^{2(k+1)}}{\sqrt{k+2}} x^{k+1}}{(-1)^k \frac{4^{2k}}{\sqrt{k+1}} x^k} \right| = \lim_{k \to \infty} 16 \sqrt{\frac{k+1}{k+2}} |x| = 16|x|
\]

Thus the radius of convergence is \( r = 1/16 \). The power series converges absolutely on the interval \((-1/16, 1/16)\).

When \( x = -1/16 \),

\[
\sum_{k=0}^{\infty} (-1)^k \frac{4^{2k}}{\sqrt{k+1}} x^k = \sum_{k=0}^{\infty} (-1)^k \frac{4^{2k}}{\sqrt{k+1}} (-1/16)^k = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}}
\]

which diverges (use the Limit Comparison Test with \( b_k = 1/\sqrt{k} \)). When \( x = 1/16 \),

\[
\sum_{k=0}^{\infty} (-1)^k \frac{4^{2k}}{\sqrt{k+1}} x^k = \sum_{k=0}^{\infty} (-1)^k \frac{4^{2k}}{\sqrt{k+1}} (1/16)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k+1}}
\]

which converges by the Alternating Series Test.

Thus the interval of convergence is \((-1/16, 1/16)\).
3. (10 points) Consider the parametric equations:

\[ x = t - 2 \sin t \]
\[ y = 1 - 2 \cos t \]

Find the slope of the tangent line to the graph of the parametric equations at \( t = 2\pi/3 \).

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2 \sin t}{1 - 2 \cos t}
\]

Thus when \( t = 2\pi/3 \),

\[ m = \frac{2 \sin 2\pi/3}{1 - 2 \cos 2\pi/3} = \frac{\sqrt{3}}{2}. \]

4. (14 points) Find a Taylor series for \( f(x) = \ln(2 + x) \) about \( c = 0 \).

Let

\[ f'(x) = \frac{1}{2 + x} \]
\[ = \frac{1}{2} \frac{1}{1 + x/2} \]
\[ = \frac{1}{2} \frac{1}{1 - (-x/2)} \]
\[ = \sum_{k=0}^{\infty} \frac{1}{2} \left( -\frac{x}{2} \right)^k \]
\[ = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} x^k \text{ if } |x| < 2. \]

Integrating term-by-term we have

\[ f(x) = \sum_{k=0}^{\infty} \int \frac{(-1)^k}{2^{k+1}} x^k \, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}(k+1)} x^{k+1}. \]
5. (8 points each) Consider the function $f(x) = \ln(\cos x)$.

(a) Find the Taylor polynomial of degree 2 about $c = \pi/6$ for $f(x)$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$f^{(k)}(x)$</th>
<th>$f^{(k)}(\pi/6)$</th>
<th>$\frac{f^{(k)}(\pi/6)}{k!}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\ln(\cos x)$</td>
<td>$\ln(\sqrt{3}/2)$</td>
<td>$\ln(\sqrt{3}/2)$</td>
</tr>
<tr>
<td>1</td>
<td>$-\tan x$</td>
<td>$-1/\sqrt{3}$</td>
<td>$-1/\sqrt{3}$</td>
</tr>
<tr>
<td>2</td>
<td>$-\sec^2 x$</td>
<td>$-4/3$</td>
<td>$-2/3$</td>
</tr>
<tr>
<td>3</td>
<td>$-2\sec^2 x \tan x$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$$P_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(\pi/6)}{k!} (x - \frac{\pi}{6})^k = \ln \left( \frac{\sqrt{3}}{2} \right) - \frac{1}{\sqrt{3}} (x - \frac{\pi}{6}) - \frac{2}{3} \left( x - \frac{\pi}{6} \right)^2$$

(b) Find the second Taylor remainder for $f(x)$ about $c = \pi/6$.

$$R_2(x) = \frac{f^{(3)}(z)}{3!} \left( x - \frac{\pi}{6} \right)^3 = -\sec^2 z \tan z \left( x - \frac{\pi}{6} \right)^3$$

where $z$ is between $x$ and $\pi/6$. 
6. (8 points) Find an equation in rectangular coordinates that has the same graph as the polar coordinate equation \( r = 2 \cos \theta + 3 \sin \theta \).

\[
\begin{align*}
  r^2 &= 2r \cos \theta + 3r \sin \theta \\
  x^2 + y^2 &= 2x + 3y \\
  x^2 - 2x + y^2 - 3y &= 0 \\
  x^2 - 2x + 1 + y^2 - 3y + \frac{9}{4} &= 1 + \frac{9}{4} \\
  (x - 1)^2 + (y - \frac{3}{2})^2 &= \frac{13}{4}
\end{align*}
\]

7. (12 points) Consider the function \( f(x) = \ln(1 + \sin x) \). Suppose the third Taylor remainder for \( f(x) \) about \( c = 0 \) is

\[
R_3(x) = \frac{-2 + \sin z}{24(\cos(z/2) + \sin(z/2))^4} x^4,
\]

where \( z \) lies between \( x \) and 0. What is the maximum error in using \( P_3(x) \) to approximate \( f(\pi/2) \)?

\[
\begin{align*}
\text{Error} &= |R_3(\pi/2)| \\
&= \left| \frac{-2 + \sin z}{24(\cos(z/2) + \sin(z/2))^4} \left( \frac{\pi}{2} \right)^4 \right| \\
&\leq \max_{0 \leq z \leq \pi/2} \left| \frac{-2 + \sin z}{24(\cos(z/2) + \sin(z/2))^4} \right| \left( \frac{\pi^4}{16} \right) \\
&= \frac{1}{12} \cdot \frac{\pi^4}{16} \\
&= \frac{\pi^4}{192} \\
&\approx 0.507339
\end{align*}
\]
8. (14 points) Consider the parametric equations:

\[
x = \cos^3 t \\
y = \sin^3 t
\]

Find the exact arclength (no approximations) of the graph of the parametric equations for \(0 \leq t \leq \pi/2\).

\[
s = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

\[
= \int_0^{\pi/2} \sqrt{(3 \cos^2 t \sin t)^2 + (3 \sin^2 t \cos t)^2} \, dt
\]

\[
= \int_0^{\pi/2} \sqrt{9 \cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t} \, dt
\]

\[
= 3 \int_0^{\pi/2} \sin t \cos t \sqrt{\cos^2 t + \sin^2 t} \, dt
\]

\[
= 3 \int_0^{\pi/2} \sin t \cos t \, dt
\]

\[
= 3 \int_0^1 u \, du
\]

\[
= \frac{3}{2}
\]