Please answer the following questions. Show all work and answers on your own paper. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank.

1. (6 points) Find the derivative of \( f(x) = x \sinh 2x - \frac{1}{2} \cosh 2x \). Please simplify your result.

\[
f'(x) = \sinh 2x + 2x \cosh 2x - \sinh 2x = 2x \cosh 2x
\]

2. (8 points) Find the volume generated when the region bounded by the given curves is rotated about the \( x \)-axis.

\[y = 4 - x^2 \quad \text{and} \quad y = 2 - x\]

The curves intersect when

\[
4 - x^2 = 2 - x
\]

\[
(4 - x^2) - (2 - x) = 0
\]

\[
(2 + x)(2 - x) - (2 - x) =
\]

\[
(2 - x)[(2 + x) - 1] =
\]

\[
(2 - x)(1 + x) = 0
\]
which implies \( x = -1 \) and \( x = 2 \). The volume of the solid of revolution can be found using the method of disks.

\[
V = \pi \int_{-1}^{2} \left( (4 - x^2)^2 - (2 - x)^2 \right) \, dx
\]

\[
= \pi \int_{-1}^{2} \left( (16 - 8x^2 + x^4) - (4 - 4x + x^2) \right) \, dx
\]

\[
= \pi \int_{-1}^{2} (12 + 4x - 9x^2 + x^4) \, dx
\]

\[
= \pi \left( 12x + 2x^2 - 3x^3 + \frac{1}{5} x^5 \right) \bigg|_{-1}^{2}
\]

\[
= \pi \left( 24 + 8 - 24 + \frac{32}{5} \right) - \pi \left( -12 + 2 + \frac{1}{5} \right)
\]

\[
= \pi \left( 8 + \frac{32}{5} + 7 + \frac{1}{5} \right)
\]

\[
= \pi \left( 15 + \frac{33}{5} \right)
\]

\[
= \frac{108}{5} \pi
\]

\[
\approx 67.8584
\]

3. (8 points) Find the radius and interval of convergence of the series

\[
\sum_{k=1}^{\infty} \frac{(x - 2)^k}{k^2}.
\]

We will apply the Ratio Test for absolute convergence.

\[
\lim_{k \to \infty} \left| \frac{(x - 2)^{k+1}}{(x - 2)^k} \right| = \lim_{k \to \infty} \left| \frac{x - 2}{k} \right| = |x - 2| < 1
\]

Thus the radius of convergence is \( r = 1 \). The power series converges absolutely in the interval \( 1 < x < 3 \). When \( x = 1 \) the power series becomes

\[
\sum_{k=1}^{\infty} \frac{(1 - 2)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}
\]

which converges by the Alternating Series Test. When \( x = 3 \) the power series becomes

\[
\sum_{k=1}^{\infty} \frac{(3 - 2)^k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}
\]

which converges as a \( p \)-series. Thus the interval of convergence for the power series is \( 1 \leq x \leq 3 \).
4. (6 points each) Evaluate the following integrals.

(a) $\int x \sin x \, dx$

Using integration by parts with

\[
\begin{align*}
   u &= x & v &= -\cos x \\
   du &= dx & dv &= \sin x \, dx
\end{align*}
\]

yields

\[
\begin{align*}
   \int x \sin x \, dx &= -x \cos x - \int (-\cos x) \, dx \\
   &= -x \cos x + \int \cos x \, dx \\
   &= -x \cos x + \sin x + C.
\end{align*}
\]

(b) $\int \frac{1}{1 + 4x^2} \, dx$

We will use a trigonometric substitution. Let $2x = \tan \theta$ and $dx = \frac{1}{2} \sec^2 \theta \, d\theta$.

\[
\begin{align*}
   \int \frac{1}{1 + 4x^2} \, dx &= \int \frac{1}{1 + \tan^2 \theta} \cdot \frac{1}{2} \sec^2 \theta \, d\theta \\
   &= \frac{1}{2} \int \frac{1}{\sec^2 \theta} \, \sec^2 \theta \, d\theta \\
   &= \frac{1}{2} \int 1 \, d\theta \\
   &= \frac{1}{2} \theta + C \\
   &= \frac{1}{2} \tan^{-1}(2x) + C
\end{align*}
\]

(c) $\int_{2}^{\infty} \frac{1}{(1 + x)^2} \, dx$

This is an improper integral.

\[
\begin{align*}
   \int_{2}^{\infty} \frac{1}{(1 + x)^2} \, dx &= \lim_{M \to \infty} \int_{2}^{M} \frac{1}{(1 + x)^2} \, dx \\
   &= \lim_{M \to \infty} \left[ -\frac{1}{1 + x} \right]_{2}^{M} \\
   &= \lim_{M \to \infty} \left( -\frac{1}{1 + M} + \frac{1}{1 + 2} \right) \\
   &= \frac{1}{3}
\end{align*}
\]
5. (6 points) Find the Cartesian coordinate form of the polar coordinate equation \( r = 2a \cos \theta \), where \( a \) is a positive constant.

\[
\begin{align*}
\frac{r}{2} &= a \cos \theta \\
\frac{r^2}{4} &= a r \cos \theta \\
x^2 + y^2 &= 2ax \\
x^2 - 2ax + y^2 &= 0 \\
x^2 - 2ax + a^2 + y^2 &= a^2 \\
(x - a)^2 + y^2 &= a^2
\end{align*}
\]

6. (6 points each) Evaluate the following limits.

(a) \( \lim_{x \to 0} \frac{\sec x - 1}{x^2} \)

This limit is indeterminate of the form 0/0. We will use l'Hôpital’s rule.

\[
\begin{align*}
\lim_{x \to 0} \frac{\sec x - 1}{x^2} &= \lim_{x \to 0} \frac{\sec x \tan x}{2x} & \text{(indeterminate 0/0)} \\
&= \lim_{x \to 0} \frac{\sec x \tan^2 x + \sec^3 x}{2} \\
&= \frac{0 + 1}{2} \\
&= \frac{1}{2}
\end{align*}
\]

(b) \( \lim_{x \to 0} \frac{\sin x^2}{x \sin x} \)

This limit is indeterminate of the form 0/0. We will use l'Hôpital’s rule.

\[
\begin{align*}
\lim_{x \to 0} \frac{\sin x^2}{x \sin x} &= \lim_{x \to 0} \frac{2x \cos x^2}{\sin x + x \cos x} & \text{(indeterminate 0/0)} \\
&= \lim_{x \to 0} \frac{2 \cos x^2 - 4x^2 \sin x^2}{2x + \cos x - x \sin x} \\
&= \frac{2 - 0}{1 + 1 - 0} \\
&= 1
\end{align*}
\]
7. (10 points) Find the Taylor polynomial of degree four \( P_4(x) \) for \( f(x) = \sec x \) about \( c = \pi/4 \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( f^{(k)}(x) )</th>
<th>( f^{(k)}(\pi/4) )</th>
<th>( f^{(k)}(\pi/4) ) \ ( k! )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \sec x )</td>
<td>( \sqrt{2} )</td>
<td>( \sqrt{2} ) \ ( 2 )</td>
</tr>
<tr>
<td>1</td>
<td>( \sec x \tan x )</td>
<td>( \sqrt{2} )</td>
<td>( \sqrt{2} ) \ ( 2 )</td>
</tr>
<tr>
<td>2</td>
<td>( \sec x \tan^2 x + \sec^3 x )</td>
<td>( 3\sqrt{2} )</td>
<td>( \frac{3\sqrt{2}}{2} ) \ ( 2 )</td>
</tr>
<tr>
<td>3</td>
<td>( \sec x \tan^3 x + 5 \sec x \tan x )</td>
<td>( 11\sqrt{2} )</td>
<td>( \frac{11\sqrt{2}}{6} ) \ ( 6 )</td>
</tr>
<tr>
<td>4</td>
<td>( \sec x \tan^4 x + 18 \sec^3 x \tan^2 x + 5 \sec^5 x )</td>
<td>( 57\sqrt{2} )</td>
<td>( \frac{19\sqrt{2}}{8} ) \ ( 8 )</td>
</tr>
</tbody>
</table>

Thus

\[
P_4(x) = \sqrt{2} + \sqrt{2} \left( x - \frac{\pi}{4} \right) + \frac{3\sqrt{2}}{2} \left( x - \frac{\pi}{4} \right)^2 + \frac{11\sqrt{2}}{6} \left( x - \frac{\pi}{4} \right)^3 + \frac{19\sqrt{2}}{8} \left( x - \frac{\pi}{4} \right)^4.
\]

8. (8 points) Find the distance traveled between \( t = 0 \) and \( t = \pi/2 \) by a particle whose position at time \( t \) is given by the parametric equations

\[
\begin{align*}
x &= \sin^2 t \\
y &= \cos^2 t
\end{align*}
\]

The arc length of the parametric curve is

\[
s = \int_0^{\pi/2} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt
\]

\[
= \int_0^{\pi/2} \sqrt{4 \cos^2 t \sin^2 t + 4 \cos^2 t \sin^2 t} \, dt
\]

\[
= \int_0^{\pi/2} \sqrt{8 \cos^2 t \sin^2 t} \, dt
\]

\[
= 2\sqrt{2} \int_0^{\pi/2} \cos t \sin t \, dt.
\]

We may integrate by substitution letting \( u = \sin t \) and \( du = \cos t \, dt \).

\[
s = 2\sqrt{2} \int_0^{\pi/2} \cos t \sin t \, dt = 2\sqrt{2} \left[ \frac{1}{2} u^2 \right]_0^{\pi/2} = \sqrt{2}
\]
9. (6 points each) Determine whether the following series converge absolutely, converge conditionally, or diverge. State the names of any convergence or divergence tests you use.

(a) \[ \sum_{k=1}^{\infty} \frac{(-1)^k \sin k}{k^2} \]

Let \( a_k = \left| (-1)^k \frac{\sin k}{k^2} \right| \) and \( b_k = \frac{1}{k^2} \). Then for all \( k \in \mathbb{N} \),

\[
0 \leq a_k \leq b_k.
\]

The series

\[ \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2} \]

is a convergent \( p \)-series \( (p = 2 > 1) \). Thus the series

\[ \sum_{k=1}^{\infty} \frac{(-1)^k \sin k}{k^2} \]

is absolutely convergent by the comparison test.

(b) \[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 3 + k}{5 + k} \]

Since

\[
\lim_{k \to \infty} \frac{(-1)^{k+1} 3 + k}{5 + k}
\]

does not exist, this series diverges according to the \( k \)th-term test.

(c) \[ \sum_{k=1}^{\infty} \frac{10^k}{k^{10}}. \]

We will apply the Ratio test.

\[
\lim_{k \to \infty} \frac{\frac{10^{k+1}}{(k+1)^{10}}}{\frac{10^k}{k^{10}}} = \lim_{k \to \infty} \left( \frac{k}{k+1} \right)^{10} \frac{10^{k+1}}{10^k} = 10 \lim_{k \to \infty} \left( \frac{k}{k+1} \right)^{10} = 10 \cdot 10 > 1
\]

The series diverges according to the Ratio test.
10. (6 points) Find the area that is inside the circle \( r = 1 \) and outside the cardioid \( r = 1 - \cos \theta \). Please include a plot of both curves on the same set of axes and label the points of intersection of the curves.

The curves intersect at the solutions of the equation

\[
\begin{align*}
1 &= 1 - \cos \theta \\
0 &= \cos \theta
\end{align*}
\]

which implies \( \theta = -\pi/2 \) and \( \theta = \pi/2 \). The desired area is then

\[
A = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left( 1^2 - (1 - \cos \theta)^2 \right) d\theta \\
= \int_{0}^{\pi/2} \left( 1^2 - (1 - \cos \theta)^2 \right) d\theta \quad \text{(by symmetry)} \\
= \int_{0}^{\pi/2} \left( 1 - (1 - 2 \cos \theta + \cos^2 \theta) \right) d\theta \\
= \int_{0}^{\pi/2} \left( 2 \cos \theta - \cos^2 \theta \right) d\theta \\
= 2 \int_{0}^{\pi/2} \cos \theta \, d\theta - \frac{1}{2} \int_{0}^{\pi/2} (1 + \cos 2\theta) \, d\theta \\
= 2 \sin \theta \bigg|_{0}^{\pi/2} - \frac{1}{2} \left( \theta + \frac{1}{2} \sin 2\theta \right) \bigg|_{0}^{\pi/2} \\
= 2 - \frac{1}{2} \left( \frac{\pi}{2} + 0 \right) \\
= 2 - \frac{\pi}{4} \approx 1.2146.
\]