Motion in Space
MATH 311, Calculus III

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Suppose the position vector of a moving object is given by

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then the tangent vector to the motion is
\[ \mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle, \]
and the magnitude of the tangent vector is
\[ \|\mathbf{r}'(t)\| = \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2 + \left[h'(t)\right]^2}. \]
Recall the arc length of a parametric curve in three dimensions is

\[ s(t) = \int_{t_0}^{t} \sqrt{[f'(u)]^2 + [g'(u)]^2 + [h'(u)]^2} \, du = \int_{t_0}^{t} \|r'(u)\| \, du. \]
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The FTC, Part II implies

\[ s'(t) = \|r'(t)\| \]

so \( \|r'(t)\| \) is the **speed** of the moving object.
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Consequently, \( r'(t) \) is the velocity vector and \( r''(t) \) is the acceleration vector.
Find the velocity and acceleration vectors of an object whose position vector is

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\[ \mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 2t^2 + 1, 3t^2 - 1, e^t \right\rangle \]

\[ \mathbf{a}(t) = \mathbf{r}''(t) = \left\langle 4t, 6t, e^t \right\rangle \]
The acceleration vector of a moving object is

\[ \mathbf{a}(t) = \left\langle \frac{t}{6}, \sin t, 0 \right\rangle, \]

while its initial velocity is \( \mathbf{v}(0) = \langle 2, 0, 3 \rangle \) and its initial position is \( \mathbf{r}(0) = \langle 0, 0, 1 \rangle \). Find the velocity and position vectors as a function of \( t \) for this object.
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\[
v(t) - v(0) = \int_0^t \left\langle \frac{s}{6}, \sin s, 0 \right\rangle \, ds
\]

\[
v(t) = \left\langle 2, 0, 3 \right\rangle + \left\langle \frac{t^2}{12}, 1 - \cos t, 0 \right\rangle
\]

\[
= \left\langle 2 + \frac{t^2}{12}, 1 - \cos t, 3 \right\rangle
\]
Finding the Position Vector

\[
\mathbf{r}(t) - \mathbf{r}(0) = \int_0^t \left\langle 2 + \frac{s^2}{12}, 1 - \cos s, 3 \right\rangle ds
\]

\[
\mathbf{r}(t) = \left\langle 0, 0, 1 \right\rangle + \left\langle 2t + \frac{t^3}{36}, t - \sin t, 3t \right\rangle
\]

\[
= \left\langle 2t + \frac{t^3}{36}, t - \sin t, 1 + 3t \right\rangle
\]
One of the most fundamental physical laws is **Newton’s second law of motion** which states that the force vector acting on an object is the product of the object’s mass (a scalar) and the object’s acceleration vector. This is stated concisely as $\mathbf{F} = m\mathbf{a}$. 

**Newton’s Second Law**
Find the force acting on an object moving along an elliptical path

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
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Let the position vector of the object be

\[ \mathbf{r}(t) = \langle a \cos \omega t, b \sin \omega t \rangle \]

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then
\[ \mathbf{F}(t) = m \mathbf{a}(t) = m \mathbf{r}''(t) = -m \omega^2 \langle a \cos t, b \sin t \rangle = -m \omega^2 \mathbf{r}(t). \]
Centripetal Motion

\[ r(t) = \langle a \cos \omega t, b \sin \omega t \rangle \]
Suppose an object is launched at an angle $\theta$ with respect to the horizontal from a height $h$. The object is given an initial speed of $s_0$. Find the position vector describing the path the object takes.
Suppose an object is launched at an angle $\theta$ with respect to the horizontal from a height $h$. The object is given an initial speed of $s_0$. Find the position vector describing the path the object takes. Treating this as motion in the $xy$-plane, we have an acceleration vector $\mathbf{a}(t) = \langle 0, -g \rangle$ and initial condition vectors

\[
\mathbf{v}_0 = s_0 \langle \cos \theta, \sin \theta \rangle \quad \text{and} \quad \mathbf{r}_0 = \langle 0, h \rangle.
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Suppose an object is launched at an angle \( \theta \) with respect to the horizontal from a height \( h \). The object is given an initial speed of \( s_0 \). Find the position vector describing the path the object takes. Treating this as motion in the \( xy \)-plane, we have an acceleration vector \( \mathbf{a}(t) = \langle 0, -g \rangle \) and initial condition vectors

\[
\mathbf{v}_0 = s_0 \langle \cos \theta, \sin \theta \rangle \quad \text{and} \quad \mathbf{r}_0 = \langle 0, h \rangle.
\]

Thus the velocity and position vectors are

\[
\mathbf{v}(t) = \langle s_0 \cos \theta, s_0 \sin \theta - gt \rangle
\]

\[
\mathbf{r}(t) = \langle (s_0 \cos \theta)t, h + (s_0 \sin \theta)t - \frac{1}{2}gt^2 \rangle.
\]
We can adapt Newton’s second law for spinning objects.

\( \tau \)  torque (scalar, \( \tau = \| \tau \| \))

\( I \)  moment of inertia, measure of the force required to start an object rotating

\( \theta \)  angle of displacement

\( \omega \)  angular velocity

\( \alpha \)  angular acceleration

\[ \alpha(t) = \omega'(t) = \theta''(t) \]
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- $I$ moment of inertia, measure of the force required to start an object rotating
- $\theta$ angle of displacement
- $\omega$ angular velocity
- $\alpha$ angular acceleration

\[
\alpha(t) = \omega'(t) = \theta''(t)
\]

\[
\tau = I\alpha
\]
A merry-go-round of radius 6 feet and moment of inertia $I = 12$ rotates at 5 radians per second. Find the constant force applied tangent to the edge of the merry-go-round needed to stop the merry-go-round in 3 seconds.
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\[
5 = \omega(3) - \omega(0) = \int_0^3 \alpha \, dt = 3\alpha
\]

implies \( \alpha = 5/3 \).

\[
\tau = (12)(5/3) = 20 = \| \mathbf{r} \times \mathbf{F} \| = \| \mathbf{r} \| \| \mathbf{F} \| = (6)\| \mathbf{F} \|
\]

which implies \( \| \mathbf{F} \| = 10/3 \) pounds.
Suppose an object has mass $m$ and velocity $v$.

- The object’s **linear momentum** is $p(t) = mv(t)$.
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- The object’s **angular momentum** is $L(t) = r(t) \times mv(t)$.

**Example**

Show that torque is the derivative of angular momentum.
Suppose an object has mass \( m \) and velocity \( \mathbf{v} \).

- The object’s **linear momentum** is \( \mathbf{p}(t) = mv(t) \).
- The object’s **angular momentum** is \( \mathbf{L}(t) = \mathbf{r}(t) \times mv(t) \).

**Example**

Show that torque is the derivative of angular momentum.

\[
\mathbf{L}'(t) = \mathbf{r}'(t) \times mv(t) + \mathbf{r}(t) \times mv'(t) \\
= \mathbf{v}(t) \times mv(t) + \mathbf{r}(t) \times ma(t) \\
= \mathbf{0} + \mathbf{r}(t) \times \mathbf{F}(t) \\
= \tau(t)
\]
A projectile of mass 10 kg is launched to the east from a height of 1 m at a speed of 10 m/s. The launch angle is 45°. The projectile spins as it flies and thus is subject to a Magnus force of magnitude 2 N in the southerly direction. Find the position of the projectile, its landing location, and its speed at impact.
Assumptions:

- East is the positive $x$-direction and south is the negative $y$-direction.
- The only forces acting on the projectile are gravity and the Magnus force.
- The projectile is launched one meter above the origin on the positive $z$-axis.
Assumptions:

- East is the positive $x$-direction and south is the negative $y$-direction.
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\[
ma = F
\]
\[
= 10\langle 0, 0, -9.8 \rangle + \langle 0, -2, 0 \rangle
\]
\[
a = \langle 0, -\frac{1}{5}, -9.8 \rangle
\]

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Motion in Space
Example (2 of 3)

\[ r''(t) = \langle 0, -\frac{1}{5}, -9.8 \rangle \]

\[ r'(t) = \langle 0, -\frac{t}{5}, -9.8t \rangle + v(0) \]

where \( v(0) = 10\langle \cos 45^\circ, 0, \sin 45^\circ \rangle = \langle 5\sqrt{2}, 0, 5\sqrt{2} \rangle. \]
\begin{align*}
r''(t) &= \langle 0, -\frac{1}{5}, -9.8 \rangle \\
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\end{align*}

where \( v(0) = 10\langle \cos 45^\circ, 0, \sin 45^\circ \rangle = \langle 5\sqrt{2}, 0, 5\sqrt{2} \rangle \).

\begin{align*}
r'(t) &= \langle 5\sqrt{2}, -\frac{t}{5}, 5\sqrt{2} - 9.8t \rangle \\
r(t) &= \langle 5\sqrt{2}t, -\frac{t^2}{10}, 5\sqrt{2}t - 4.9t^2 \rangle + r(0)
\end{align*}

where \( r(0) = \langle 0, 0, 1 \rangle \).
\[ r''(t) = \langle 0, -\frac{1}{5}, -9.8 \rangle \]
\[ r'(t) = \langle 0, -\frac{t}{5}, -9.8t \rangle + v(0) \]

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\[ r'(t) = \langle 5\sqrt{2}, -\frac{t}{5}, 5\sqrt{2} - 9.8t \rangle \]
\[ r(t) = \langle 5\sqrt{2}t, -\frac{t^2}{10}, 5\sqrt{2}t - 4.9t^2 \rangle + r(0) \]

where \( r(0) = \langle 0, 0, 1 \rangle \).

\[ r(t) = \langle 5\sqrt{2}t, -\frac{t^2}{10}, 1 + 5\sqrt{2}t - 4.9t^2 \rangle \]
Since

\[ \mathbf{r}(t) = \langle 5\sqrt{2}t, -\frac{t^2}{10}, 1 + 5\sqrt{2}t - 4.9t^2 \rangle \]

the projectile lands when

\[ 1 + 5\sqrt{2}t - 4.9t^2 = 0 \implies t \approx 1.57283, \]

which implies

\[ \mathbf{r}(1.57283) \approx \langle 11.1216, -0.247379, 0 \rangle \]

\[ \|\mathbf{r}'(1.57283)\| \approx 10.9407 \]
Homework

- Read Section 11.3.
- Exercises: 1–55 odd.