Tangent and Normal Vectors
MATH 311, *Calculus III*

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When an observer is traveling along with a moving point, for example the passengers in an airplane, it can be useful to have a right-handed coordinate system travel with the observer.

Q: How do we create a moving coordinate system?
A: We already have the unit tangent vector

$$ T(t) = \frac{r'(t)}{||r'(t)||} $$

which is in the same direction as the motion. We need only two more vectors perpendicular to it.
Principal Unit Normal Vector

Definition

The **principal unit normal vector** $N(t)$ is a unit vector having the same direction as $T'(t)$ and is defined by

$$N(t) = \frac{T'(t)}{\|T'(t)\|}.$$
Definition

The principal unit normal vector \( \mathbf{N}(t) \) is a unit vector having the same direction as \( \mathbf{T}'(t) \) and is defined by

\[
\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.
\]

Note: \( \mathbf{N}(t) \neq \frac{\mathbf{r}''(t)}{\|\mathbf{r}''(t)\|} \)
By the Chain Rule, \( T'(t) = \frac{dT}{dt} = \frac{dT}{ds} \frac{ds}{dt} \) so

\[
\mathbf{N}(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{\frac{dT}{ds} \frac{ds}{dt}}{\left\| \frac{dT}{ds} \right\|} \frac{\frac{dT}{ds}}{\frac{ds}{dt}}
\]

\[
= \frac{\frac{dT}{ds}}{\left\| \frac{dT}{ds} \right\|} \quad \text{(since } \|r'(t)\| = \frac{ds}{dt} > 0)\]

\[
= \frac{1}{\kappa} \frac{dT}{ds} \quad \text{(since } \kappa = \left\| \frac{dT}{ds} \right\|)\]

\( \frac{dT}{ds} \) points in the direction \( T \) is turning and thus \( \mathbf{N}(t) \) always points to the concave side of the curve.
Find the unit tangent and principal unit normal vectors for the following vector-valued function.

\[ r(t) = \langle t, t^3 \rangle \]
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\[ r(t) = \langle t, t^3 \rangle \]

\[
T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{\sqrt{1 + 9t^4}} \langle 1, 3t^2 \rangle
\]

Remark: in \( V_2 \), vectors \( \langle a, b \rangle \) and \( \langle -b, a \rangle \) are always orthogonal.
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\[ T'(t) = \frac{6t}{(1 + 9t^4)^{3/2}} \langle -3t^2, 1 \rangle \]

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\[ \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{2}}{\sqrt{3 - \cos 2t}} \langle \cos t, -\sin t, -\sin t \rangle \]
Example (2 of 2)

Find the unit tangent and principal unit normal vectors for the following vector-valued function.

\[ \mathbf{r}(t) = \langle \sin t, \cos t, \cos t \rangle \]

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\[ \mathbf{T}'(t) = \frac{-2\sqrt{2}}{(3 - \cos 2t)^{3/2}} \langle 2 \sin t, \cos t, \cos t \rangle \]
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\[ \mathbf{N}(t) = -\frac{1}{\sqrt{3 - \cos 2t}} \langle 2\sin t, \cos t, \cos t \rangle \]
So far we have a pair of orthogonal vectors defined along the path of motion. We can get the third orthogonal vector from the cross product.

\[
\text{Binormal Vector} 
\]

**Definition**

The binormal vector \( B(t) \) is

\[
B(t) = T(t) \times N(t)
\]

**Note:**

\[
\|B(t)\| = \|T(t) \times N(t)\| = \|T(t)\| \|N(t)\| \sin \frac{\pi}{2} = 1.
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**Definition**

The triple of unit vectors $T(t)$, $N(t)$, and $B(t)$ forms a moving frame of reference called the **TNB frame** or **moving trihedral**.
Tangent and Normal Vectors
Find the binormal vector for the vector-valued function,

\[ r(t) = \langle \sin t, \cos t, \cos t \rangle \]
Example

Find the binormal vector for the vector-valued function,

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We have already found:

\[
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\[ B(t) = \frac{1}{\sqrt{2}} \langle 0, 1, -1 \rangle \]
Osculating Circle

**Definition**

For a curve $C$ in the $xy$-plane, defined by the vector-valued function $\mathbf{r}(t)$, if the curvature $\kappa \neq 0$ when $t = t_0$, we define the **radius of curvature** to be $1/\kappa$. The **center of curvature** is the terminal point of the vector $\mathbf{r}(t) + \frac{1}{\kappa} \mathbf{N}(t)$ (when $\kappa \neq 0$). The **osculating circle** or **circle of curvature** is the circle whose center is the center of curvature and whose radius is the radius of curvature (when $\kappa \neq 0$).
Tangent and Normal Vectors

Illustration
Find the center of curvature and the radius of curvature for the osculating circle to the curve \( r(t) = \langle t, t^3 \rangle \) when \( t = 1 \).
Example

Find the center of curvature and the radius of curvature for the osculating circle to the curve \( \mathbf{r}(t) = \langle t, t^3 \rangle \) when \( t = 1 \).

\[
\kappa = \frac{|f''(t)|}{(1 + [f'(t)]^2)^{3/2}} = \frac{6|t|}{(1 + 9t^4)^{3/2}}
\]

\[
\mathbf{N}(t) = \frac{1}{\sqrt{1 + 9t^4}} \langle -3t^2, 1 \rangle
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\[
\text{radius} = \frac{1}{\kappa(1)} = \frac{5\sqrt{10}}{3}
\]
Find the center of curvature and the radius of curvature for the osculating circle to the curve \( \mathbf{r}(t) = \langle t, t^3 \rangle \) when \( t = 1 \).

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radius \( = \frac{1}{\kappa(1)} = \frac{5\sqrt{10}}{3} \)

center \( = \mathbf{r}(1) + \frac{1}{\kappa(1)} \mathbf{N}(1) = \left( -4, \frac{8}{3} \right) \)
The unit tangent and principal unit normal vectors can explain the forces which work to stabilize and destabilize an object as it moves on a path.
The **unit tangent** and **principal unit normal** vectors can explain the forces which work to stabilize and destabilize an object as it moves on a path.

The velocity of an object moving along a path described by $\mathbf{r}(t)$ is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \|\mathbf{r}'(t)\| \mathbf{T}(t) = \frac{ds}{dt} \mathbf{T}(t).$$
The **unit tangent** and **principal unit normal** vectors can explain the forces which work to stabilize and destabilize an object as it moves on a path.

The velocity of an object moving along a path described by \( \mathbf{r}(t) \) is

\[
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\]

The acceleration is given by

\[
\mathbf{a}(t) = \frac{d}{dt} \left( \frac{ds}{dt} \mathbf{T}(t) \right) = \frac{d^2s}{dt^2} \mathbf{T}(t) + \frac{ds}{dt} \mathbf{T}'(t).
\]
\[ \mathbf{a}(t) = \frac{d^2 s}{dt^2} \mathbf{T}(t) + \frac{ds}{dt} \mathbf{T}'(t) \]

\[ = \frac{d^2 s}{dt^2} \mathbf{T}(t) + \frac{ds}{dt} \| \mathbf{T}'(t) \| \mathbf{N}(t) \]

\[ = \frac{d^2 s}{dt^2} \mathbf{T}(t) + \frac{ds}{dt} \left\| \frac{d\mathbf{T}}{ds} \frac{ds}{dt} \right\| \mathbf{N}(t) \]

\[ = \frac{d^2 s}{dt^2} \mathbf{T}(t) + \left( \frac{ds}{dt} \right)^2 \left\| \frac{d\mathbf{T}}{ds} \right\| \mathbf{N}(t) \]

\[ = \frac{d^2 s}{dt^2} \mathbf{T}(t) + \kappa \left( \frac{ds}{dt} \right)^2 \mathbf{N}(t) \]
\[ a(t) = \frac{d^2 s}{dt^2} T(t) + \kappa \left( \frac{ds}{dt} \right)^2 N(t) \]

\[ \frac{d^2 s}{dt^2} \]: is called the **tangential component of acceleration** and is denoted \( a_T = \frac{d^2 s}{dt^2} \).

\[ \kappa \left( \frac{ds}{dt} \right)^2 \]: is called the **normal component of acceleration** and is denoted \( a_N = \kappa \left( \frac{ds}{dt} \right)^2 \).
\[ a(t) = \frac{d^2 s}{dt^2} \mathbf{T}(t) + \kappa \left( \frac{ds}{dt} \right)^2 \mathbf{N}(t) \]

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By Newton’s Second Law of Motion

\[ \mathbf{F}(t) = m\mathbf{a}(t) = m \frac{d^2 s}{dt^2} \mathbf{T}(t) + m\kappa \left( \frac{ds}{dt} \right)^2 \mathbf{N}(t). \]
Express the acceleration of the object traveling along the path described by $\mathbf{r}(t) = \langle \cos t, \sin t, \sin 3t \rangle$ in terms of the tangential and normal components.

**Strategy:**
- Find $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$.
- Find $a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}\|\mathbf{r}'(t)\|$.
- Since $\|\mathbf{a}(t)\|^2 = a_T^2 + a_N^2$ then find $a_N = \sqrt{\|\mathbf{a}(t)\|^2 - a_T^2}$.
Express the acceleration of the object traveling along the path described by \( \mathbf{r}(t) = \langle \cos t, \sin t, \sin 3t \rangle \) in terms of the tangential and normal components.

**Strategy:**

1. Find \( \frac{ds}{dt} = \| \mathbf{r}'(t) \| \).
2. Find \( a_T = \frac{d^2s}{dt^2} = \frac{d}{dt} \| \mathbf{r}'(t) \| \).
3. Since \( \| \mathbf{a}(t) \|^2 = a_T^2 + a_N^2 \) then find \( a_N = \sqrt{\| \mathbf{a}(t) \|^2 - a_T^2} \).
\[ \mathbf{r}(t) = \langle \cos t, \sin t, \sin 3t \rangle \]
Example (2 of 2)

\[ \mathbf{r}(t) = \langle \cos t, \sin t, \sin 3t \rangle \]

Steps:

\[ \frac{ds}{dt} = \sqrt{\frac{11}{2} + \frac{9}{2} \cos 6t} \]
Example (2 of 2)

\[ \mathbf{r}(t) = \langle \cos t, \sin t, \sin 3t \rangle \]

**Steps:**

\[
\frac{ds}{dt} = \sqrt{\frac{11}{2} + \frac{9}{2} \cos 6t}
\]

\[
a_T = \frac{-27 \sin 6t}{2 \sqrt{\frac{11}{2} + \frac{9}{2} \cos 6t}}.
\]
Example (2 of 2)

\[ \mathbf{r}(t) = \langle \cos t, \sin t, \sin 3t \rangle \]

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\[ \frac{ds}{dt} = \sqrt{\frac{11}{2} + \frac{9}{2} \cos 6t} \]

\[ a_T = \frac{-27 \sin 6t}{2 \sqrt{\frac{11}{2} + \frac{9}{2} \cos 6t}}. \]

\[ a_N = 2 \sqrt{\frac{23 - 18 \cos 6t}{11 + 9 \cos 6t}} \]
Tangential components of acceleration are shown in green.
Normal components of acceleration are shown in red.
Application: Finding Curvature

\[ a(t) = \frac{d^2 s}{dt^2} T(t) + \kappa \left( \frac{ds}{dt} \right)^2 N(t) \]

\[ a(t) \times T(t) = \frac{d^2 s}{dt^2} T(t) \times T(t) + \kappa \left( \frac{ds}{dt} \right)^2 N(t) \times T(t) = 0 \]

\[ \| a(t) \times T(t) \| = \kappa \left( \frac{ds}{dt} \right)^2 \]

\[ \left\| \frac{r''(t) \times r'(t)}{\| r'(t) \|} \right\| = \kappa \| r'(t) \|^2 \]

\[ \left\| \frac{r''(t) \times r'(t)}{\| r'(t) \|^3} \right\| = \kappa \]
Application: Finding the Binormal Vector

\[ \mathbf{a}(t) = \frac{d^2 s}{dt^2} \mathbf{T}(t) + \kappa \left( \frac{ds}{dt} \right)^2 \mathbf{N}(t) \]

\[ \mathbf{T}(t) \times \mathbf{a}(t) = \kappa \left( \frac{ds}{dt} \right)^2 \mathbf{B}(t) \]

\[ \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\| \mathbf{r}'(t) \|} = \frac{\| \mathbf{r}'(t) \times \mathbf{r}''(t) \|}{\| \mathbf{r}'(t) \|^{3/2}} \| \mathbf{r}'(t) \| \mathbf{B}(t) \]

\[ \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\| \mathbf{r}'(t) \times \mathbf{r}''(t) \|} = \mathbf{B}(t) \]
Kepler’s Laws of Planetary Motion

1. Each planet follows an elliptical orbit, with the sun at one focus.

2. The line segment joining the sun to a planet sweeps out equal areas in equal times.

3. If $T$ is the time required for a given planet to make one orbit of the sun and if the length of the major axes of its elliptical orbit is $2a$, then $T^2 = ka^3$ for some constant $k$. 

We can derive Kepler’s three laws from two of Newton’s Laws of Motion.
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We can derive Kepler’s three laws from two of Newton’s Laws of Motion.
Assumptions

1. The sun is at the origin and the planet is at the terminal point of vector $\mathbf{r}(t)$.

2. The gravitational force of attraction has a magnitude which is proportional to the inverse of the square of the distance separating the sun and the planet.

$$F(t) = m a(t) = -\frac{GmM}{\|\mathbf{r}(t)\|^2}$$ (Newton's 2nd Law)
Assumptions

1. The sun is at the origin and the planet is at the terminal point of vector $r(t)$.

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\[ F(t) = ma(t) = -\frac{GmM}{\|r(t)\|^2} r(t) \] (Newton's 2nd Law)
Assumptions

1. The sun is at the origin and the planet is at the terminal point of vector \( \mathbf{r}(t) \).

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\[
\mathbf{F}(t) = ma(t) \quad \text{(Newton’s 2nd Law)}
\]

\[
= - \frac{GmM}{\|r(t)\|^2} \frac{r(t)}{\|r(t)\|}
\]
Let $r = \|r(t)\|$ and note that $\frac{r(t)}{\|r(t)\|}$ is a unit vector, call it $u$.

\[
F = ma = -\frac{GmM}{r^2}u
\]

which implies
Let $r = \|r(t)\|$ and note that $\frac{r(t)}{\|r(t)\|}$ is a unit vector, call it $\mathbf{u}$.

$$F = ma = -\frac{GmM}{r^2}\mathbf{u}$$

which implies

$$a = -\frac{GM}{r^2}\mathbf{u}.$$  

**Note:** this says the acceleration is always in the opposite direction of $r(t)$, toward the sun.
Consider

\[ \frac{d}{dt}(r(t) \times v(t)) = r'(t) \times v(t) + r(t) \times v'(t) \]

\[ = v(t) \times v(t) + r(t) \times a(t) \]

\[ = 0 + 0 = 0 \]
Consider

\[
\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{v}(t)) = \mathbf{r}'(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{v}'(t)
\]

\[
= \mathbf{v}(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{a}(t)
\]

\[
= 0 + 0 = 0
\]

Hence \(\mathbf{r} \times \mathbf{v} = \mathbf{c}\) a constant vector. This implies the orbit of the planet must lie in plane.

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Tangent and Normal Vectors
Kepler’s First Law

Each planet follows an elliptical orbit, with the sun at one focus.
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Assume the planet’s orbit is in the $xy$-plane so that $\mathbf{c}$ is parallel to the $z$-axis.
Kepler’s First Law

Each planet follows an elliptical orbit, with the sun at one focus.

Assume the planet’s orbit is in the $xy$-plane so that $c$ is parallel to the $z$-axis.

Since $\mathbf{r}(t) = r\mathbf{u}$, then $\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d}{dt}(r\mathbf{u}) = \frac{dr}{dt}\mathbf{u} + r\frac{d\mathbf{u}}{dt}$
\[ \mathbf{c} = \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \left( \frac{dr}{dt} \mathbf{u} + r \frac{d\mathbf{u}}{dt} \right) = \mathbf{r} \times \frac{dr}{dt} \mathbf{u} + \mathbf{r} \times r \frac{d\mathbf{u}}{dt} = \frac{dr}{dt} \mathbf{r} \times \mathbf{u} + r^2 \left( \mathbf{u} \times \frac{d\mathbf{u}}{dt} \right) = r^2 \left( \mathbf{u} \times \frac{d\mathbf{u}}{dt} \right) \]
Kepler’s First Law (continued)

\[ \mathbf{a} \times \mathbf{c} = \mathbf{a} \times r^2 \left( \mathbf{u} \times \frac{d\mathbf{u}}{dt} \right) \]

\[ = -\frac{GM}{r^2} \mathbf{u} \times r^2 \left( \mathbf{u} \times \frac{d\mathbf{u}}{dt} \right) \]

\[ = -GM \mathbf{u} \times \left( \mathbf{u} \times \frac{d\mathbf{u}}{dt} \right) \]

\[ = -GM \begin{bmatrix} \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} \end{bmatrix} \mathbf{u} - \left( \mathbf{u} \cdot \mathbf{u} \right) \frac{d\mathbf{u}}{dt} \]

\[ = GM \frac{d\mathbf{u}}{dt} \]
Kepler’s First Law (continued)

\[ GM \frac{du}{dt} = a \times c \]

\[ = \frac{dv}{dt} \times c \]

\[ = \frac{d}{dt}(v \times c) \quad \text{(since } c \text{ is constant)} \]

\[ GMu + b = v \times c \quad \text{(where } b \text{ is constant)} \]

**Note:** since \( v \times c \) is orthogonal to \( c \) then \( v \times c \) is a vector in the \( xy \)-plane. Vector \( u \) is in the \( xy \)-plane, so \( b \) must also be in the \( xy \)-plane. We can choose \( b \) to be parallel to \( \mathbf{i} \).
Kepler’s First Law (continued)

\[ \left\| \mathbf{c} \right\|^2 = \mathbf{c} \cdot \mathbf{c} \]
\[ = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{c} \]
\[ = \mathbf{r} \cdot (\mathbf{v} \times \mathbf{c}) \]
\[ = r \mathbf{u} \cdot (GM \mathbf{u} + \mathbf{b}) \]
\[ = rGM \mathbf{u} \cdot \mathbf{u} + r \mathbf{u} \cdot \mathbf{b} \]
\[ = 1 \]
\[ = rGM + r \| \mathbf{b} \| \cos \theta \]
Kepler’s First Law (continued)

\[ \|c\|^2 = c \cdot c \]
\[ = (r \times v) \cdot c \]
\[ = r \cdot (v \times c) \]
\[ = ru \cdot (GMu + b) \]
\[ = rGMu \cdot u + ru \cdot b \]
\[ = rGM + r\|b\| \cos \theta \]

**Remark:** Angle \( \theta \) is the angle between \( r \) and \( i \).
Letting $c = \|c\|$ and $b = \|b\|$ we may solve for $r$:

\[
  r = \frac{c^2}{GM + b \cos \theta} \quad = \quad \frac{ed}{1 + e \cos \theta}
\]

where $e = \frac{b}{GM}$ and $d = \frac{c^2}{b}$.
Letting $c = \|c\|$ and $b = \|b\|$ we may solve for $r$:

$$
\begin{align*}
    r &= \frac{c^2}{GM + b \cos \theta} \\
    &= \frac{ed}{1 + e \cos \theta}
\end{align*}
$$

where $e = \frac{b}{GM}$ and $d = \frac{c^2}{b}$.

This is the equation of an ellipse in polar coordinates!
Read Section 11.5.
Exercises: 1–29 odd, 39, 40