Diagonal, Triangular, and Symmetric Matrices
MATH 322, Linear Algebra I

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Some matrices have special organization or structure:

- Diagonal matrices
- Triangular matrices
- Symmetric matrices

Later we will study the conditions under which general matrices are equivalent to these special forms.
Diagonal Matrices

**Definition**

A **diagonal matrix** is a square matrix in which all entries off the diagonal are zero.
Diagonal Matrices

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General form:

\[
D = \begin{bmatrix}
    d_1 & 0 & \cdots & 0 \\
    0 & d_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & d_n
\end{bmatrix}
\]
**Inverses of Diagonal Matrices**

**Inverses:** if $D$ is a diagonal matrix and $(D)_{ii} \neq 0$ for $i = 1, 2, \ldots, n$ then

$$D^{-1} = \begin{bmatrix}
1/d_1 & 0 & \cdots & 0 \\
0 & 1/d_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1/d_n
\end{bmatrix}$$
Powers of Diagonal Matrices

**Powers:** if $D$ is a diagonal matrix and $k \in \mathbb{N}$ then

\[
D^k = \begin{bmatrix}
  d_1^k & 0 & \cdots & 0 \\
  0 & d_2^k & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & d_n^k \\
\end{bmatrix}
\]
Examples

Given \( A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \), find

1. \( A^{-1} \)
2. \( A^4 \)
3. \( A^{-3} \)
Explore the effect of multiplying a generic matrix on the left or right by a diagonal matrix.

\[
\begin{bmatrix}
  d_1 & 0 \\
  0 & d_2
\end{bmatrix}
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23}
\end{bmatrix}
= \\
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
d_1 & 0 \\
0 & d_2
\end{bmatrix}
= \]
Multiplication by Diagonal Matrices

Explore the effect of multiplying a generic matrix on the left or right by a diagonal matrix.

\[
\begin{bmatrix}
  d_1 & 0 \\
  0 & d_2 \\
\end{bmatrix} \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
\end{bmatrix} = \begin{bmatrix}
  d_1 a_{11} & d_1 a_{12} & d_1 a_{13} \\
  d_2 a_{21} & d_2 a_{22} & d_2 a_{23} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22} \\
\end{bmatrix} \begin{bmatrix}
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Multiplication by Diagonal Matrices

Explore the effect of multiplying a generic matrix on the left or right by a diagonal matrix.

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\]

\[
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  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
  d_1 & 0 \\
  0 & d_2
\end{bmatrix}
= 
\begin{bmatrix}
  d_1 a_{11} & d_2 a_{12} \\
  d_1 a_{21} & d_2 a_{22}
\end{bmatrix}
\]

- \(DA\) is a matrix whose \(i^{th}\) row is the \(i^{th}\) row of \(A\) multiplied by \(d_i\).
- \(AD\) is a matrix whose \(j^{th}\) column is the \(j^{th}\) column of \(A\) multiplied by \(d_j\).
Definition
A square matrix $U$ is upper triangular if $(U)_{ij} = 0$ if $i > j$.

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$
Definition
A square matrix $L$ is **lower triangular** if $(L)_{ij} = 0$ if $i < j$.

$$L = \begin{bmatrix}
a_{11} & 0 & \cdots & 0 \\
a_{21} & a_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}$$
Diagonal matrices are both upper *and* lower triangular.
Diagonal matrices are both upper \emph{and} lower triangular.

A square matrix in row-echelon form is upper triangular.
Comments

- Diagonal matrices are both upper *and* lower triangular.
- A square matrix in row-echelon form is upper triangular.
- A square matrix is upper triangular if and only if the $i^{th}$ row starts with at least $i - 1$ zeros.
Comments

- Diagonal matrices are both upper and lower triangular.
- A square matrix in row-echelon form is upper triangular.
- A square matrix is upper triangular if and only if the $i^{th}$ row starts with at least $i - 1$ zeros.
- A square matrix is lower triangular if and only if the $j^{th}$ column starts with at least $j - 1$ zeros.
Properties of Triangular Matrices

Theorem

1. The transpose of an upper (lower) triangular matrix is lower (upper) triangular.
2. The product of upper (lower) triangular matrices is upper (lower) triangular.
3. A triangular matrix is invertible if and only if all its diagonal entries are nonzero.
4. The inverse of a upper (lower) triangular matrix is upper (lower) triangular.

The proof of 1 is left as an exercise.
Properties of Triangular Matrices

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Proof of 2

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be upper triangular $n \times n$ matrices and let $C = AB$. 

By the matrix multiplication algorithm $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.

Suppose $j < i$ (i.e., $c_{ij}$ is below the diagonal), then

$$c_{ij} = \sum_{k=1}^{j} a_{ik}b_{kj} + \sum_{k=j+1}^{n} a_{ik}b_{kj}.$$ 

Since $a_{ik}$ and $b_{kj}$ are both zero for $k > j$, we have

$$c_{ij} = \sum_{k=1}^{j} a_{ik}b_{kj} + \sum_{k=j+1}^{n} (0) = 0.$$ 

This implies $C$ is upper triangular.
Proof of 2

- Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be upper triangular $n \times n$ matrices and let $C = AB$.
- By the matrix multiplication algorithm

\[ c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}. \]
Proof of 2

- Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be upper triangular $n \times n$ matrices and let $C = AB$.
- By the matrix multiplication algorithm

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$ 
- Suppose $j < i$ ($c_{ij}$ is below the diagonal), then

$$c_{ij} = \sum_{k=1}^{j} a_{ik}b_{kj} + \sum_{k=j+1}^{n} a_{ik}b_{kj}$$

$$= \sum_{k=1}^{j} (0)b_{kj} + \sum_{k=1j+1}^{n} a_{ik}(0)$$

$$= 0$$

which implies $C$ is upper triangular.
Examples

Given $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 5 & 2 & 3 \end{bmatrix}$ and $M = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & -2 & -3 \end{bmatrix}$, find

1. $L^{-1}$
2. $LM$
Symmetric Matrices

Definition
Matrix $A$ is a **symmetric matrix** if $A = A^T$, i.e.

$$(A)_{ij} = (A)_{ji} \text{ for all } i,j.$$

General form:

$$A = \begin{bmatrix}
        a_{11} & a_{12} & \cdots & a_{1n} \\
        a_{12} & a_{22} & \cdots & a_{2n} \\
        \vdots & \vdots & \ddots & \vdots \\
        a_{1n} & a_{2n} & \cdots & a_{nn}
\end{bmatrix}$$
Properties of Symmetric Matrices

Theorem
If $A$ and $B$ are symmetric matrices of the same size and if $k$ is a scalar, then

1. $A^T$ is symmetric.
2. $A + B$ and $A - B$ are symmetric.
3. $kA$ is symmetric.
Products of Symmetric Matrices

Suppose $A$ and $B$ are symmetric matrices of the same size and consider

$$(AB)^T =$$
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$$(AB)^T = B^T A^T =$$
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Suppose $A$ and $B$ are symmetric matrices of the same size and consider

$$(AB)^T = B^T A^T = BA \neq AB$$
Suppose $A$ and $B$ are symmetric matrices of the same size and consider

$$(AB)^T = B^T A^T = BA = AB$$

The product of two symmetric matrices is symmetric if and only if the matrices commute, i.e., $AB = BA$. 
Inverses of Symmetric Matrices

Theorem
If $A$ is symmetric and invertible, then $A^{-1}$ is symmetric.
Inverses of Symmetric Matrices

**Theorem**
*If A is symmetric and invertible, then \( A^{-1} \) is symmetric.*

**Proof.**
Assume \( A \) is symmetric and invertible, then

\[
(A^{-1})^T = (A^T)^{-1} = A^{-1}
\]

which implies \( A^{-1} \) is symmetric.

\(\square\)
Products $AA^T$ and $A^T A$

In a variety of situations we encounter matrix products of the form $AA^T$ or $A^T A$. 
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1. what is the size of $AA^T$?
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1. what is the size of $AA^T$?
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3. what is $(AA^T)^T$?
Products $AA^T$ and $A^TA$

In a variety of situations we encounter matrix products of the form $AA^T$ or $A^TA$.

Suppose $A$ is an $m \times n$ matrix, then

1. what is the size of $AA^T$?
2. what is the size of $A^TA$?
3. what is $(AA^T)^T$?
4. what is $(A^TA)^T$?
Theorem

If $A$ is an invertible matrix then $AA^T$ and $A^T A$ are also invertible.
Invertible Matrices

Theorem
If $A$ is an invertible matrix then $AA^T$ and $A^TA$ are also invertible.

Proof.
If $A$ is invertible, then $A^T$ is invertible and the product of invertible matrices is invertible.
Homework

- Read Section 1.7
- Exercises: 1–6, 8, 12, 16–19, 22, 27, 37, 41