7) 

a) \((-1,2) \cdot (6,3) = -6 + 6 = 0 \quad \Rightarrow \text{ orthogonal}\)

\[
\frac{(-1,2)}{\|(-1,2)\|} = \frac{(-1,2)}{\sqrt{1+4}} = \left( -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right), \quad \frac{(6,3)}{\|(6,3)\|} = \left( \frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \right)
\]

b) \((1,0,-1) \cdot (2,0,2) = 2 + 0 - 2 = 0 \quad \Rightarrow \text{orthogonal}\)

\[(1,0,-1) \cdot (0,5,0) = 0 + 0 + 0 = 0 \quad \Rightarrow \text{orthogonal}\]

\[(2,0,2) \cdot (0,5,0) = 0 + 0 + 0 = 0 \quad \Rightarrow \text{orthogonal}\]

\[
\frac{(1,0,-1)}{\|(1,0,-1)\|} = \left( \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2} \right)
\]

\[
\frac{(2,0,2)}{\|(2,0,2)\|} = \left( \frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \right)
\]

\[
\frac{(0,5,0)}{\|(0,5,0)\|} = (0,1,0)
\]

c) \((\frac{1}{5}, \frac{1}{5}, \frac{1}{5}) \cdot (-\frac{1}{2}, \frac{1}{2}, 0) = -\frac{1}{10} + \frac{1}{10} + 0 = 0 \quad \Rightarrow \text{orthogonal}\)

\[(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}) \cdot (\frac{1}{5}, \frac{1}{5}, -\frac{2}{5}) = \frac{1}{25} + \frac{1}{25} - \frac{2}{25} = 0 \quad \Rightarrow \text{orthogonal}\]

\[(-\frac{1}{2}, \frac{1}{2}, 0) \cdot (\frac{1}{5}, \frac{1}{5}, -\frac{2}{5}) = -\frac{1}{10} + \frac{1}{10} + 0 = 0 \quad \Rightarrow \text{orthogonal}\]

\[
\frac{(\frac{1}{5}, \frac{1}{5}, \frac{1}{5})}{\|(\frac{1}{5}, \frac{1}{5}, \frac{1}{5})\|} = \left( \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right)
\]

\[
\frac{(-\frac{1}{2}, \frac{1}{2}, 0)}{\|(-\frac{1}{2}, \frac{1}{2}, 0)\|} = \left( -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right)
\]

\[
\frac{(\frac{1}{5}, \frac{1}{5}, -\frac{2}{5})}{\|(\frac{1}{5}, \frac{1}{5}, -\frac{2}{5})\|} = \left( \frac{\sqrt{10}}{5}, \frac{\sqrt{10}}{5}, -\frac{\sqrt{10}}{5} \right)
\]
9) \[ \mathbf{\nu}_1 = \begin{pmatrix} -\frac{3}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}, \quad \mathbf{\nu}_2 = \begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \\ 0 \end{pmatrix}, \quad \mathbf{\nu}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

\[ \mathbf{\nu}_1 \cdot \mathbf{\nu}_2 = -\frac{12}{25} + \frac{12}{25} + 0 = 0 \]
\[ \mathbf{\nu}_1 \cdot \mathbf{\nu}_3 = 0 + 0 + 0 = 0 \]
\[ \mathbf{\nu}_2 \cdot \mathbf{\nu}_3 = 0 + 0 + 0 = 0 \]
\[ \Rightarrow \text{orthonormal} \]

\[ \mathbf{\nu}_1 \cdot \mathbf{\nu}_1 = 1 \]
\[ \mathbf{\nu}_2 \cdot \mathbf{\nu}_2 = 1 \]
\[ \mathbf{\nu}_3 \cdot \mathbf{\nu}_3 = 1 \]
\[ \Rightarrow \text{orthonormal} \]

a) \[ \mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \]
\[ \mathbf{u} = (\mathbf{u} \cdot \mathbf{\nu}_1) \mathbf{\nu}_1 + (\mathbf{u} \cdot \mathbf{\nu}_2) \mathbf{\nu}_2 + (\mathbf{u} \cdot \mathbf{\nu}_3) \mathbf{\nu}_3 \]
\[ \mathbf{u} = -\frac{7}{5} \mathbf{\nu}_1 + \frac{1}{5} \mathbf{\nu}_2 + 2 \mathbf{\nu}_3 \]

b) \[ \mathbf{u} = \begin{pmatrix} 3 \\ -7 \\ 4 \end{pmatrix} \]
\[ \mathbf{u} = (\mathbf{u} \cdot \mathbf{\nu}_1) \mathbf{\nu}_1 + (\mathbf{u} \cdot \mathbf{\nu}_2) \mathbf{\nu}_2 + (\mathbf{u} \cdot \mathbf{\nu}_3) \mathbf{\nu}_3 \]
\[ \mathbf{u} = -\frac{37}{5} \mathbf{\nu}_1 - \frac{9}{5} \mathbf{\nu}_2 + 4 \mathbf{\nu}_3 \]

c) \[ \mathbf{u} = \begin{pmatrix} \frac{1}{7} \\ -\frac{3}{7} \\ \frac{5}{7} \end{pmatrix} \]
\[ \mathbf{u} = (\mathbf{u} \cdot \mathbf{\nu}_1) \mathbf{\nu}_1 + (\mathbf{u} \cdot \mathbf{\nu}_2) \mathbf{\nu}_2 + (\mathbf{u} \cdot \mathbf{\nu}_3) \mathbf{\nu}_3 \]
\[ \mathbf{u} = \frac{3}{7} \mathbf{\nu}_1 - \frac{1}{7} \mathbf{\nu}_2 + \frac{5}{7} \mathbf{\nu}_3 \]
14) a) \( (u)_s = (-1, 2, 1, 3) \)
\( (v)_s = (0, -3, 1, 5) \)
\( (w)_s = (-2, -4, 3, 1) \)

\[ \|u\| = \sqrt{(-1)^2 + (2)^2 + (1)^2 + (3)^2} = \sqrt{15} \]

\[ \|v - w\| = \sqrt{(0 - (-2))^2 + (-3 - (4))^2 + (1 - 3)^2 + (5 - 1)^2} = \sqrt{25} = 5 \]

\[ \|v + w\| = \sqrt{(0 + 2)^2 + (-3 + 4)^2 + (1 + 3)^2 + (5 + 1)^2} = \sqrt{105} \]

\[ \langle v, w \rangle = (0)(-2) + (-3)(-4) + (1)(3) + (5)(1) = 20 \]

b) \( (u)_s = (0, 0, -1, -1) \)
\( (v)_s = (5, 5, -2, -2) \)
\( (w)_s = (3, 0, -3, 0) \)

\[ \|u\| = \sqrt{0^2 + 0^2 + (-1)^2 + (-1)^2} = \sqrt{2} \]

\[ \|v - w\| = \sqrt{(5 - 3)^2 + (5 - 0)^2 + (-2 - (-3))^2 + (-2 - 0)^2} = \sqrt{34} \]

\[ \|v + w\| = \sqrt{(5 + 3)^2 + (5 + 0)^2 + (-2 + (-3))^2 + (-2 + 0)^2} = \sqrt{118} \]

\[ \langle v, w \rangle = (5)(3) + (5)(0) + (-2)(-3) + (-2)(0) = 21 \]
17. (a) \[ \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]

\[ \hat{\mathbf{S}}_1 = \hat{\mathbf{u}}_1 \]

\[ \hat{\mathbf{S}}_2 : \mathbf{u}_2 = \frac{\langle \mathbf{u}_2, \hat{\mathbf{S}}_1 \rangle \hat{\mathbf{S}}_1}{\| \hat{\mathbf{S}}_1 \|^2} = \mathbf{u}_2 \]

\[ \hat{\mathbf{S}}_3 : \mathbf{u}_3 = \frac{\langle \mathbf{u}_3, \hat{\mathbf{S}}_1 \rangle \hat{\mathbf{S}}_1 - \langle \mathbf{u}_3, \hat{\mathbf{S}}_2 \rangle \hat{\mathbf{S}}_2}{\| \hat{\mathbf{S}}_2 \|^2} \]

\[ \hat{\mathbf{S}}_3 = \begin{bmatrix} 1/6 \\ 1/6 \\ -1/3 \end{bmatrix} \]

\[ \hat{\mathbf{S}}_1 = \frac{\hat{\mathbf{S}}_1}{\| \hat{\mathbf{S}}_1 \|} = \begin{bmatrix} \sqrt{5}/3 \\ \sqrt{5}/3 \\ \sqrt{5}/3 \end{bmatrix} \]

\[ \hat{\mathbf{S}}_2 = \frac{\hat{\mathbf{S}}_2}{\| \hat{\mathbf{S}}_2 \|} = \begin{bmatrix} -\sqrt{15}/2 \\ \sqrt{15}/2 \\ 0 \end{bmatrix} \]

\[ \hat{\mathbf{S}}_3 = \frac{\hat{\mathbf{S}}_3}{\| \hat{\mathbf{S}}_3 \|} = \begin{bmatrix} \sqrt{6}/6 \\ \sqrt{6}/6 \\ -\sqrt{6}/3 \end{bmatrix} \]
24e) \[
\begin{bmatrix}
1 & 2 \\
1 & 1 \\
0 & 3 \\
\end{bmatrix}
\]

Purify Gram-Schmidt orthogonalization on the columns of the matrix.

\[
\hat{v}_1 = \begin{bmatrix}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} \\
0
\end{bmatrix}
\]

\[
\hat{v}_2 = \begin{bmatrix}
2 \\
1 \\
3
\end{bmatrix} - \frac{\langle \hat{u}_3, \hat{v}_1 \rangle}{\| \hat{v}_1 \|^2} \hat{v}_1 = \begin{bmatrix}
2 \\
1 \\
3
\end{bmatrix} - \frac{10}{14} \begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} \\
-\frac{1}{2} \\
3
\end{bmatrix}
\]

\[
\hat{v}_3 = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} - \frac{\langle \hat{u}_3, \hat{v}_1 \rangle}{\| \hat{v}_1 \|^2} \hat{v}_1 - \frac{\langle \hat{u}_3, \hat{v}_2 \rangle}{\| \hat{v}_2 \|^2} \hat{v}_2 = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} - \frac{10}{14} \begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix} - \frac{-\frac{3}{19}}{19} \begin{bmatrix}
\frac{1}{2} \\
-\frac{1}{2} \\
3
\end{bmatrix} = \begin{bmatrix}
-\frac{3}{19} \\
\frac{3}{19} \\
\frac{1}{19}
\end{bmatrix}
\]

\[
\hat{q}_1 = \frac{\hat{v}_1}{\| \hat{v}_1 \|} = \begin{bmatrix}
\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} \\
0
\end{bmatrix}
\]

\[
\hat{q}_2 = \frac{\hat{v}_2}{\| \hat{v}_2 \|} = \begin{bmatrix}
\frac{\sqrt{3}/3}{3} \\
\frac{-\sqrt{3}/3}{3} \\
\frac{3}{3}
\end{bmatrix}
\]

\[
\hat{q}_3 = \frac{\hat{v}_3}{\| \hat{v}_3 \|} = \begin{bmatrix}
\frac{-3/19}{19} \\
\frac{3/19}{19} \\
\frac{1/19}{19}
\end{bmatrix}
\]

\[
Q = \begin{bmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{3}/3}{3} & -\frac{3/19}{19} \\
\frac{\sqrt{2}}{2} & \frac{-\sqrt{3}/3}{3} & \frac{3/19}{19} \\
0 & \frac{3/19}{19} & \frac{1/19}{19}
\end{bmatrix}
\]
\[ R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \langle u_3, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \langle u_3, q_2 \rangle \\ 0 & 0 & \langle u_3, q_3 \rangle \end{bmatrix} = \begin{bmatrix} 1 & 3\sqrt{2}/2 & \sqrt{2} \\ 0 & \sqrt{3}/2 & 3\sqrt{3}/2 \\ 0 & 0 & \sqrt{19}/19 \end{bmatrix} \]
20) \( \mathcal{B} = \{v_1, v_2, \ldots, v_n \} \) is an orthonormal basis for an inner product space \( V \).

Claim: \( \forall \omega \in V \), then \( \|\omega\|^2 = \langle \omega, v_1 \rangle^2 + \langle \omega, v_2 \rangle^2 + \ldots + \langle \omega, v_n \rangle^2 \)

Proof:

Since \( \mathcal{B} \) is an orthonormal basis, then

\[
\omega = \langle \omega, v_1 \rangle v_1 + \langle \omega, v_2 \rangle v_2 + \ldots + \langle \omega, v_n \rangle v_n
\]

\[
\|\omega\|^2 = \langle \omega, \omega \rangle = \langle \omega, v_1 \rangle^2 + \langle \omega, v_2 \rangle^2 + \ldots + \langle \omega, v_n \rangle^2
\]

\[
= \langle \omega, v_1 \rangle \sum_{k=1}^{n} \langle v_k, v_1 \rangle \langle v_k, v_k \rangle
\]

\[
= \langle \omega, v_1 \rangle \sum_{k=1}^{n} \langle v_k, v_1 \rangle \langle v_k, v_k \rangle
\]

\[
= \langle \omega, v_1 \rangle \sum_{k=1}^{n} \langle v_k, v_1 \rangle \langle v_k, v_k \rangle
\]

\[
= \langle \omega, v_1 \rangle \sum_{k=1}^{n} \langle v_k, v_1 \rangle \langle v_k, v_k \rangle
\]

\[
= \langle \omega, v_1 \rangle^2 + \langle \omega, v_2 \rangle^2 + \ldots + \langle \omega, v_n \rangle^2
\]
27) **Claim:** The linear dependence of \( \{u_1, u_2, \ldots, u_r \} \) ensures that \( v_3 \not\perp 0 \).

**Proof:**

(By contradiction) Assume \( v_3 \perp 0 \).

\[
0 = v_3 = \frac{\langle u_3, v_1 \rangle v_1}{\|v_1\|^2} - \frac{\langle u_3, v_2 \rangle v_2}{\|v_2\|^2}
\]

\[
\Rightarrow u_3 = \frac{\langle u_3, v_1 \rangle v_1}{\|v_1\|^2} + \frac{\langle u_3, v_2 \rangle v_2}{\|v_2\|^2} - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2
\]

\[
= \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} \left( v_2 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2 \right)
\]

\[
u_3 = \left( \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2 \|v_2\|^2} \right) v_1 + \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2.
\]

i.e. \( u_3 \in \text{span}\{u_1, u_2\} \Rightarrow \{u_1, u_2, \ldots, u_r \} \) is not linearly independent. This contradicts our assumption.

28) **Claim:** The diagonal elements \( \theta_i \) are non-zero.

**Proof:**

Consider the diagonal matrix \( \langle u_i, \xi_i \rangle \).

\( \langle u_i, \xi_i \rangle = 0 \Rightarrow \xi_i \) is orthogonal to \( u_1, u_2, \ldots, u_i \).

Thus \( \xi_i \) is spanned by \( \{u_1, u_2, \ldots, u_i\} \) which contradicts the definition of \( \xi_i \). Thus \( \langle u_i, \xi_i \rangle \neq 0 \).
21) \( S = \{1, x, x^2\} \)

\[ u_1 = 1 \]

\[ u_2 = x - \frac{x}{\|1\|^2} = x - \frac{\frac{1}{2} x^2 dx}{\frac{1}{2}} = x - \frac{1}{2} = x \]

\[ u_3 = x^2 - \frac{x^2}{\|1\|^2} - \left( x \right) (0) = x^2 - \frac{1}{3} \]

\[ u_1 = \frac{\|u_1\|}{\|v_1\|} = \frac{\sqrt{2}}{2} \]

\[ u_2 = \frac{\|u_2\|}{\|v_2\|} = \frac{x}{\|\frac{1}{2} x^2 dx\|} = \frac{\sqrt{2}}{2} = \frac{x}{\sqrt{2/3}} = \frac{x}{2} \]

\[ u_3 = \frac{\|u_3\|}{\|v_3\|} = \frac{x^2 - \frac{1}{3}}{\left( \frac{1}{2} (x^2 - \frac{1}{3}) dx \right)^{1/2}} = 3 \sqrt{\frac{5}{8}} (x^2 - \frac{1}{3}) \]