Orthogonal Matrices; Change of Basis
MATH 322, Linear Algebra I

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Objectives

Sometimes choosing the appropriate basis when working with a vector space can make a specific problem easier. In this lesson we will

- discuss a class of matrices whose inverses can be found by transposition, and
- use them in a variety of applications.
Orthogonal Matrices

Definition
A square matrix $A$ with the property that $A^{-1} = A^T$ is said to be an orthogonal matrix.
Orthogonal Matrices

Definition
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Remark: A square matrix $A$ is orthogonal if and only if either $AA^T = I$ or $A^TA = I$. 
Example (1 of 2)

Verify that $A = \frac{1}{3} \begin{bmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ 2 & 1 & 2 \end{bmatrix}$ is an orthogonal matrix.
Verify that the matrix which rotates a vector through angle $\theta$ about the origin is an orthogonal matrix.
Equivalent Statements

Theorem

For an $n \times n$ matrix $A$ the following statements are equivalent.

1. $A$ is orthogonal.
2. The row vectors of $A$ form an orthonormal set in $\mathbb{R}^n$ with the Euclidean inner product.
3. The column vectors of $A$ form an orthonormal set in $\mathbb{R}^n$ with the Euclidean inner product.
Proof (1) ⇐⇒ (2)

- Assume $A$ is an orthogonal matrix, let $r_i$ be the $i$th row of $A$ and let $c_j$ be the $j$th column of $A^T$.

- Consider the product

$$AA^T = \begin{bmatrix}
  r_1 \cdot c_1 & r_1 \cdot c_2 & \cdots & r_1 \cdot c_n \\
  r_2 \cdot c_1 & r_2 \cdot c_2 & \cdots & r_2 \cdot c_n \\
  \vdots & \vdots & \ddots & \vdots \\
  r_n \cdot c_1 & r_n \cdot c_2 & \cdots & r_n \cdot c_n
\end{bmatrix} = \begin{bmatrix}
  r_1 \cdot r_1 & r_1 \cdot r_2 & \cdots & r_1 \cdot r_n \\
  r_2 \cdot r_1 & r_2 \cdot r_2 & \cdots & r_2 \cdot r_n \\
  \vdots & \vdots & \ddots & \vdots \\
  r_n \cdot r_1 & r_n \cdot r_2 & \cdots & r_n \cdot r_n
\end{bmatrix}$$

- $AA^T = I$ if and only if $r_i \cdot r_i = 1$ for $i = 1, 2, \ldots, n$ and $r_i \cdot r_j = 0$ when $i \neq j$.

- $B = \{r_1, r_2, \ldots, r_n\}$ is an orthonormal set in $\mathbb{R}^n$. 
Properties of Orthogonal Matrices

Theorem
Suppose that $A$ and $B$ are $n \times n$ orthogonal matrices, then

- $A^T$ is orthogonal,
- $A^{-1}$ is orthogonal,
- $AB$ is orthogonal, and
- $\det(A) = \pm 1$. 
Orthogonal Matrices as Linear Operators

Theorem
For an $n \times n$ matrix $A$ the following statements are equivalent.

1. $A$ is orthogonal.
2. $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^n$.
3. $Ax \cdot Ay = x \cdot y$ for all $x, y \in \mathbb{R}^n$.

Remark: If $T_A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is multiplication by an orthogonal matrix $A$, then $T_A$ is called an orthogonal operator on $\mathbb{R}^n$. 
Theorem
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Remark: If $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is multiplication by an orthogonal matrix $A$, then $T$ is called an orthogonal operator on $\mathbb{R}^n$. 
Proof $(1) \implies (2) \implies (3)$

- Suppose $A$ is orthogonal, then

$$
\|Ax\| = \sqrt{Ax \cdot Ax} = \sqrt{x \cdot A^TAx} = \sqrt{x \cdot x} = \|x\|.
$$
Proof (1) \implies (2) \implies (3)

- Suppose \( A \) is orthogonal, then

\[
\|Ax\| = \sqrt{Ax \cdot Ax} = \sqrt{x \cdot A^TAx} = \sqrt{x \cdot x} = \|x\|.
\]

- Suppose \( \|Ax\| = \|x\| \) for all \( x \in \mathbb{R}^n \).

\[
Ax \cdot Ay = \frac{1}{4}\|Ax + Ay\|^2 - \frac{1}{4}\|Ax - Ay\|^2
\]

\[
= \frac{1}{4}\|A(x + y)\|^2 - \frac{1}{4}\|A(x - y)\|^2
\]

\[
= \frac{1}{4}\|x + y\|^2 - \frac{1}{4}\|x - y\|^2
\]

\[
= x \cdot y
\]
Proof (3) \implies (1)

- Suppose that $Ax \cdot Ay = x \cdot y$ for all $x, y \in \mathbb{R}^n$, then

\[
x \cdot y = x \cdot A^T Ay
\]
\[
x \cdot y - x \cdot A^T Ay = 0
\]
\[
x \cdot (I - A^TA)y = 0
\]
Proof $(3) \implies (1)$

- Suppose that $Ax \cdot Ay = x \cdot y$ for all $x, y \in \mathbb{R}^n$, then

\[
\begin{align*}
  x \cdot y &= x \cdot A^T Ay \\
  x \cdot y - x \cdot A^T Ay &= 0 \\
  x \cdot (I - A^T A)y &= 0
\end{align*}
\]

- Let $x = (I - A^T A)y$ in the last equation.

\[
\begin{align*}
  (I - A^T A)y \cdot (I - A^T A)y &= 0 \\
  (I - A^T A)y &= 0 \quad \text{(positivity axiom)}
\end{align*}
\]
Proof (3) $\iff$ (1)

- Suppose that $A \mathbf{x} \cdot A \mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then
  \[
  \mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^T A \mathbf{y} \\
  \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot A^T A \mathbf{y} = 0 \\
  \mathbf{x} \cdot (I - A^T A) \mathbf{y} = 0
  \]

- Let $\mathbf{x} = (I - A^T A) \mathbf{y}$ in the last equation.
  \[
  (I - A^T A) \mathbf{y} \cdot (I - A^T A) \mathbf{y} = 0 \\
  (I - A^T A) \mathbf{y} = 0 \quad \text{(positivity axiom)}
  \]

- Since this holds for all $\mathbf{y} \in \mathbb{R}^n$ then $I - A^T A = 0$, the $n \times n$ zero matrix. Consequently $A^T A = I$ and $A$ is orthogonal.
Recall: If $B = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space $V$ and $v \in V$ then there are scalars $k_1, k_2, \ldots, k_n$ such that

$$v = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n.$$ 

The vector of coordinates of $v$ relative to $B$ is denoted

$$(v)_B = (k_1, k_2, \ldots, k_n)$$

and the coordinate matrix of $v$ relative to $B$ is

$$[v]_B = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}. $$
Properties of a Transition Matrix

Theorem
If P is the transition matrix from \( \mathcal{B}' \) to \( \mathcal{B} \), then

1. \( P \) is invertible.
2. \( P^{-1} \) is the transition matrix from \( \mathcal{B} \) to \( \mathcal{B}' \).
Properties of a Transition Matrix

**Theorem**

*If $P$ is the transition matrix from $B'$ to $B$, then*

1. $P$ is invertible.
2. $P^{-1}$ is the transition matrix from $B$ to $B'$.

\[
[v]_B = P[v]_{B'} \quad \text{and} \quad [v]_{B'} = P^{-1}[v]_B
\]
Change of Orthonormal Basis

Theorem
If $P$ is the transition matrix from one orthonormal basis to another orthonormal basis for a finite-dimensional inner product space, then $P$ is an orthogonal matrix, i.e., $P^{-1} = P^T$. 
Proof (1 of 2)

Let $V$ is a finite-dimensional inner product space, let $\mathcal{B}$ and $\mathcal{B}'$ be two orthonormal bases for $V$, and let $P$ be the transition matrix from $\mathcal{B}'$ to $\mathcal{B}$.

To avoid confusion with the vector norm on $\mathbb{R}^n$ relative to the Euclidean inner product (denoted $\| \|$), we will denote the vector norm on $V$ relative to its inner product as $\| \|_V$. 
Proof (2 of 2)

- For any orthonormal basis for $V$ the norm of any $u \in V$ is the same as the Euclidean norm of its coordinate vector.

\[
\|u\|_V = \| (u)_{B'} \| = \| (u)_B \| = \| P(u)_{B'} \|
\]
Proof (2 of 2)

For any orthonormal basis for $V$ the norm of any $u \in V$ is the same as the Euclidean norm of its coordinate vector.

\[ \|u\|_V = \|(u)_{B'}\| = \|(u)_B\| = \|P(u)_{B'}\| \]

Let $x$ be any vector in $\mathbb{R}^n$ and $u$ be the vector in $V$ whose coordinate vector with respect to basis $B'$ is $x$.

\[ \|u\| = \|x\| = \|Px\| \]

which proves that matrix $P$ is orthogonal.
Homework

- Read Section 7.1
- Exercises: 1, 3, 5, 7, 11