Section 1.2, Exercise 11

Consider the falling object of mass 10 kg in Example 2, but assume now that the drag force is proportional to the square of the velocity.

(a) If the limiting velocity is 49 m/sec (the same as in Example 2), show that the equation of motion can be written as

\[ \frac{dv}{dt} = \frac{(49)^2 - v^2}{245}. \]

Note, in order to avoid a preponderance of minus signs we will assume that the downward (falling) direction is the positive direction. According to Newton’s Second Law of Motion, \( ma = \sum F \). There are two forces at work on the falling object: (1) gravity and (2) the drag force, thus

\[ m \frac{dv}{dt} = 9.8m - \gamma v^2 \]

where \( \gamma > 0 \) is the proportionality constant. Since the limiting velocity is 49 m/sec, then when \( v = 49 \) we know \( \frac{dv}{dt} = 0 \), thus

\[
\begin{align*}
10 \frac{dv}{dt} & = 98 - \gamma v^2 \\
0 & = 98 - \gamma (49)^2 \\
\gamma & = \frac{2}{49}.
\end{align*}
\]

Substituting this value into the equation above gives an equation of motion of the form

\[
\begin{align*}
m \frac{dv}{dt} & = 9.8m - \gamma v^2 \\
10 \frac{dv}{dt} & = 98 - \frac{2}{49} v^2 \\
\frac{dv}{dt} & = \frac{98 - \frac{2}{49} v^2}{10} \\
& = \frac{49 - \frac{1}{49} v^2}{5} \\
& = \frac{49(49 - \frac{1}{49} v^2)}{(5)(49)} \\
& = \frac{(49)^2 - v^2}{245}.
\end{align*}
\]

(b) If \( v(0) = 0 \), find an expression for \( v(t) \) at any time.
Consider the initial value problem
\[
\frac{dv}{dt} = \frac{(49)^2 - v^2}{245}, \\
v(0) = 0.
\]

We can separate variables in the ODE to obtain
\[
\frac{dv}{(49)^2 - v^2} = \frac{1}{245} \, dt,
\]

\[
\int_0^v \frac{1}{(49)^2 - s^2} \, ds = \int_0^t \frac{1}{245} \, ds
\]

To integrate the left hand side of the equation we must make use of the hyperbolic trigonometric identity
\[
1 - \tanh^2 z = \sech^2 z.
\]

Make the substitution in the integral
\[
s = 49 \tanh u, \\
ds = 49 \sech^2 u \, du,
\]
then
\[
\int_0^{\tanh^{-1} v/49} \frac{1}{(49)^2 - s^2} \, ds = \frac{t}{245},
\]

\[
\frac{49}{(49)^2 - (49)^2 \tanh^2 u} \sech^2 u \, du = \frac{t}{245},
\]

\[
\frac{1}{49} \int_0^{\tanh^{-1} v/49} \frac{1}{1 - \tanh^2 u} \sech^2 u \, du = \frac{t}{245},
\]

\[
\int_0^{\tanh^{-1} v/49} \frac{1}{\sech^2 u} \, du = \frac{49t}{245},
\]

\[
\int_0^{\tanh^{-1} v/49} 1 \, du = \frac{t}{5},
\]

\[
\tanh^{-1} \frac{v}{49} = \frac{t}{5},
\]

\[
v(t) = 49 \tanh \left( \frac{t}{5} \right).
\]

(c) Plot your solution from part (b) and the solution (26) from Example 2 on the same axes.
(d) Based on your plots in part (c), compare the effect of a quadratic drag force with that of a linear drag force.

The object subject to the quadratic drag force obtains the terminal velocity sooner than the object subject to the linear drag force.

(e) Find the distance $x(t)$ that the object falls in time $t$.

Assuming the initial position is $x(0) = 0$,

\[ v(t) = 49 \tanh \left( \frac{t}{5} \right) \]

\[ \frac{dx}{dt} = 49 \frac{\sinh \left( \frac{s}{5} \right)}{\cosh \left( \frac{s}{5} \right)} \]

\[ dx = 49 \frac{\sinh \left( \frac{s}{5} \right)}{\cosh \left( \frac{s}{5} \right)} \, dt \]

\[ \int_0^x ds = 49 \int_0^t \frac{\sinh \left( \frac{s}{5} \right)}{\cosh \left( \frac{s}{5} \right)} \, ds \]

\[ x = 49 \int_0^t \frac{\sinh \left( \frac{s}{5} \right)}{\cosh \left( \frac{s}{5} \right)} \, ds. \]

We will integrate by substitution after letting

\[ u = \cosh \left( \frac{s}{5} \right) \]

\[ 5 \, du = \sinh \left( \frac{s}{5} \right) \, ds. \]

Thus

\[ x = 49 \int_0^t \frac{\sinh \left( \frac{s}{5} \right)}{\cosh \left( \frac{s}{5} \right)} \, ds \]
\[
= 49 \int_1^{\cosh \frac{t}{5}} \frac{5}{u} \, du \\
= 245 \left( \ln \cosh \left( \frac{t}{5} \right) - \ln 1 \right) \\
\]

\[
\begin{align*}
x(t) &= 245 \ln \cosh \left( \frac{t}{5} \right).
\end{align*}
\]

(f) Find the time \( T \) it takes to fall 300 meters.

\[
\begin{align*}
300 &= 245 \ln \cosh \left( \frac{T}{5} \right) \\
\frac{60}{49} &= \ln \cosh \left( \frac{T}{5} \right) \\
e^{60/49} &= \cosh \left( \frac{T}{5} \right) \\
cosh^{-1} \left( e^{60/49} \right) &= \frac{T}{5} \\
5 \cosh^{-1} \left( e^{60/49} \right) &= T \\
9.47653 &\approx T.
\end{align*}
\]