Variation of Parameters
MATH 365 Ordinary Differential Equations

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Introduction

Previously we were introduced to the **method of undetermined coefficients** for solving certain types of linear, nonhomogeneous ODEs.

Today we take up the method of **variation of parameters** which has the advantage that

- it can be applied for any continuous, nonhomogeneous term $g(t)$,
- the result is given by a formula and requires no guesswork as to the proper form of the solution.
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Today we take up the method of **variation of parameters** which has the advantage that

- it can be applied for any continuous, nonhomogeneous term $g(t)$,
- the result is given by a formula and requires no guesswork as to the proper form of the solution.

It had the disadvantage that it may require evaluation of some difficult integrals.
Variation of Parameters

Theorem

Consider the equation

\[ y'' + p(t)y' + q(t)y = g(t) \]

where \( p, q, \) and \( g \) are continuous on an open interval \( I. \)

Suppose that \( y_1 \) and \( y_2 \) are linearly independent solutions to the corresponding homogeneous equation, then a particular solution to the nonhomogeneous equation is

\[ Y(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)} \, ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)} \, ds, \]

where \( t_0 \) is any conveniently chosen point in \( I. \)
Suppose $Y(t) = \mu_1(t)y_1(t) + \mu_2(t)y_2(t)$ is a solution to the nonhomogeneous equation. Since there are two unknowns, we may impose two conditions.

\[ Y'(t) = \mu_1'(t)y_1(t) + \mu_1(t)y_1'(t) + \mu_2'(t)y_2(t) + \mu_2(t)y_2'(t) \]
\[ = \mu_1'(t)y_1(t) + \mu_2'(t)y_2(t) + \mu_1(t)y_1'(t) + \mu_2(t)y_2'(t) \]
\[ \text{assume } = 0 \]
\[ = \mu_1(t)y_1'(t) + \mu_2(t)y_2'(t) \]

\[ Y''(t) = \mu_1'(t)y_1'(t) + \mu_1(t)y_1''(t) + \mu_2'(t)y_2'(t) + \mu_2(t)y_2''(t) \]
Proof (2 of 3)

The second condition is that $Y(t)$ must solve the nonhomogeneous ODE.

\[
g(t) = \mu_1'(t)y_1'(t) + \mu_1(t)y_1''(t) + \mu_2'(t)y_2'(t) + \mu_2(t)y_2''(t) \\
+ p(t)[\mu_1(t)y_1'(t) + \mu_2(t)y_2'(t)] \\
+ q(t)[\mu_1(t)y_1(t) + \mu_2(t)y_2(t)] \\
= \mu_1(t)[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)] \\
+ \mu_2(t)[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)] \\
= \mu_1'(t)y_1'(t) + \mu_2'(t)y_2'(t) \\
= \mu_1(t)y_1'(t) + \mu_2(t)y_2'(t)
\]
Thus we have the system of two equations for the two unknowns $\mu'_1(t)$ and $\mu'_2(t)$:

\[
0 = \mu'_1(t)y_1(t) + \mu'_2(t)y_2(t)
\]
\[
g(t) = \mu'_1(t)y'_1(t) + \mu'_2(t)y'_2(t)
\]

Using Cramer’s Rule, the solutions are

\[
\mu'_1(t) = -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)}
\]
\[
\mu'_2(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}.
\]
Example

Find the general solution to the following second order linear nonhomogeneous ODE.

$$y'' - y' - 2y = e^{-t}$$

**Note:** in this case we can verify the solution using the method of undetermined coefficients.
Example

Find the general solution to the following second order linear nonhomogeneous ODE.

\[ y'' - y' - 2y = e^{-t} \]

**Note:** in this case we can verify the solution using the method of undetermined coefficients.

\[ y_c(t) = c_1 e^{-t} + c_2 e^{2t} \]
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**Note:** in this case we can verify the solution using the method of undetermined coefficients.

\[ y_c(t) = c_1 e^{-t} + c_2 e^{2t} \]

\[ y(t) = c_1 e^{-t} + c_2 e^{2t} - \frac{1}{3} te^{-t} \]
Solution (1 of 2)

Let \( y_1(t) = e^{-t} \) and \( y_2(t) = e^{2t} \), then

\[
W(y_1, y_2)(t) = \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} = 3e^t.
\]
Let \( y_1(t) = e^{-t} \) and \( y_2(t) = e^{2t} \), then

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W(y_1, y_2)(t) = \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} = 3e^t.
\]

\[
\mu_1'(t) = -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)} = -\frac{-e^{2t}e^{-t}}{3e^t} = -\frac{1}{3}
\]

\[
\mu_1(t) = -\frac{1}{3}t
\]
Solution (1 of 2)

Let \( y_1(t) = e^{-t} \) and \( y_2(t) = e^{2t} \), then

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\]

\[
\mu_1(t) = -\frac{1}{3}t
\]

\[
\mu_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} = \frac{e^{-t}e^{-t}}{3e^t} = \frac{1}{3}e^{-3t}
\]

\[
\mu_2(t) = \frac{-1}{9}e^{-3t}
\]
The particular solution is

\[ Y(t) = \mu_1(t)y_1(t) + \mu_2(t)y_2(t) = \frac{1}{3}te^{-t} - \frac{1}{9}e^{-3t}e^{2t} \]

\[ = \frac{1}{3}te^{-t} - \frac{1}{9}e^{-t}. \]

Since the second term is a complementary solution, we may ignore it.

Thus the general solution to the nonhomogeneous equation is

\[ y(t) = c_1 e^{-t} + c_2 e^{2t} - \frac{1}{3}te^{-t}. \]
Example

Find the general solution to the following second order linear nonhomogeneous ODE.

\[ y'' + 4y = \sec^2 2t \]
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\[ y_c(t) = c_1 \cos(2t) + c_2 \sin(2t) \]
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\[ y(t) = c_1 \cos(2t) + c_2 \sin(2t) \]

\[ + \frac{1}{4} \sin(2t) \ln |\sec(2t) + \tan(2t)| - \frac{1}{4} \]
Solution (1 of 2)

Let \( y_1(t) = \cos(2t) \) and \( y_2(t) = \sin(2t) \), then

\[
W(y_1, y_2)(t) = \begin{vmatrix}
\cos(2t) & \sin(2t) \\
-2\sin(2t) & 2\cos(2t)
\end{vmatrix} = 2.
\]
Let $y_1(t) = \cos(2t)$ and $y_2(t) = \sin(2t)$, then

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} = 2.$$ 

$$\mu_1'(t) = \frac{-y_2(t)g(t)}{W(y_1, y_2)(t)} = \frac{-\sin(2t)\sec^2(2t)}{2} = -\frac{1}{2}\sec(2t)\tan(2t)$$

$$\mu_1(t) = -\frac{1}{4}\sec(2t)$$
Solution (1 of 2)

Let \(y_1(t) = \cos(2t)\) and \(y_2(t) = \sin(2t)\), then

\[
W(y_1, y_2)(t) = \begin{vmatrix}
\cos(2t) & \sin(2t) \\
-2\sin(2t) & 2\cos(2t)
\end{vmatrix} = 2.
\]

\[
\mu'_1(t) = -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)} = -\frac{-\sin(2t)\sec^2(2t)}{2} = -\frac{1}{2} \sec(2t)\tan(2t)
\]

\[
\mu_1(t) = -\frac{1}{4} \sec(2t)
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\[
\mu'_2(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} = \frac{\cos(2t)\sec^2(2t)}{2} = \frac{1}{2} \sec(2t)
\]

\[
\mu_2(t) = \frac{1}{4} \ln |\sec(2t) + \tan(2t)|
\]
The particular solution is

\[ Y(t) = \mu_1(t)y_1(t) + \mu_2(t)y_2(t) \]

\[ = -\frac{1}{4} \sec(2t) \cos(2t) + \frac{1}{4} (\sin(2t)) \ln |\sec(2t) + \tan(2t)| \]

\[ = -\frac{1}{4} + \frac{1}{4} (\sin(2t)) \ln |\sec(2t) + \tan(2t)|. \]

Thus the general solution to the nonhomogeneous equation is

\[ y(t) = c_1 \cos(2t) + c_2 \sin(2t) - \frac{1}{4} + \frac{1}{4} (\sin(2t)) \ln |\sec(2t) + \tan(2t)|. \]
Find the general solution to the following second order linear nonhomogeneous ODE.

\[ y'' + y = \cot t \]
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\[ y'' + y = \cot t \]

\[ y_c(t) = c_1 \cos t + c_2 \sin t \]

\[ y(t) = c_1 \cos t + c_2 \sin t - (\sin t) \ln | \csc t + \cot t | \]
Solution (1 of 2)

Let \( y_1(t) = \cos t \) and \( y_2(t) = \sin t \), then

\[
W(y_1, y_2)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1.
\]
Solution (1 of 2)

Let \( y_1(t) = \cos t \) and \( y_2(t) = \sin t \), then

\[
W(y_1, y_2)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1.
\]

\[
\mu'_1(t) = \frac{-y_2(t)g(t)}{W(y_1, y_2)(t)} = \frac{-\sin t \cot t}{1} = -\cos t
\]

\[
\mu_1(t) = -\sin t
\]
Solution (1 of 2)

Let $y_1(t) = \cos t$ and $y_2(t) = \sin t$, then

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\mu_1(t) = -\sin t
\]

\[
\mu'_2(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} = \frac{\cos t \cot t}{1} = \frac{\cos^2 t}{\sin t} = \frac{1 - \sin^2 t}{\sin t}
\]

\[
\mu_2(t) = \cos t - \ln |\csc t + \cot t|
\]
The particular solution is

\[ Y(t) = \mu_1(t)y_1(t) + \mu_2(t)y_2(t) \]
\[ = -\sin t \cos t + \cos t \sin t - (\sin t) \ln |\csc t + \cot t| \]
\[ = -(\sin t) \ln |\csc t + \cot t|. \]

Thus the general solution to the nonhomogeneous equation is

\[ y(t) = c_1 \cos t + c_2 \sin t - (\sin t) \ln |\csc t + \cot t|. \]
Example

Verify that \( y_1(t) = t \) and \( y_2(t) = te^t \) are a fundamental set of solutions to the homogeneous version of the nonhomogeneous ODE below. Find the general solution to the nonhomogeneous ODE.

\[
t^2 y'' - t(t + 2)y' + (t + 2)y = 2t^3
\]
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y_c(t) = c_1 t + c_2 te^t
\]
\[
y(t) = c_1 t + c_2 te^t - 2t^2
\]

Before using variation of parameters, we must put the ODE in standard form,

\[
y''' - \frac{t + 2}{t} y' + \frac{t + 2}{t^2} y = 2t.
\]
Solution (1 of 2)

Let $y_1(t) = t$ and $y_2(t) = te^t$, then

$$W(y_1, y_2)(t) = \begin{vmatrix} t & te^t \\ 1 & (t + 1)e^t \end{vmatrix} = t^2 e^t.$$
Let $y_1(t) = t$ and $y_2(t) = te^t$, then

$$W(y_1, y_2)(t) = \begin{vmatrix} t & te^t \\ 1 & (t + 1)e^t \end{vmatrix} = t^2 e^t.$$ 

$$\mu'_1(t) = \frac{-y_2(t)g(t)}{W(y_1, y_2)(t)} = \frac{-te^t(2t)}{t^2 e^t} = -2$$

$$\mu_1(t) = -2t$$
Solution (1 of 2)

Let \( y_1(t) = t \) and \( y_2(t) = te^t \), then

\[
W(y_1, y_2)(t) = \begin{vmatrix} t & te^t \\ 1 & (t + 1)e^t \end{vmatrix} = t^2 e^t.
\]

\[
\mu_1'(t) = \frac{-y_2(t)g(t)}{W(y_1, y_2)(t)} = \frac{-te^t(2t)}{t^2 e^t} = -2
\]

\[
\mu_1(t) = -2t
\]

\[
\mu_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} = \frac{t(2t)}{t^2 e^t} = 2e^{-t}
\]

\[
\mu_2(t) = -2e^{-t}
\]
The particular solution is

\[ Y(t) = \mu_1(t)y_1(t) + \mu_2(t)y_2(t) \]
\[ = -2t^2 + (-2e^{-t}te^t) \]
\[ = -2t^2 - 2t \]

Since the second term is a complementary solution, we may ignore it.

Thus the general solution to the nonhomogeneous equation is

\[ y(t) = c_1 t + c_2 te^t - 2t^2. \]
Example

Find the general solution to the following second order linear nonhomogeneous ODE.

\[ y'' + 4y' + 4y = \frac{e^{-2t}}{t^2} \]
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\[ y'' + 4y' + 4y = \frac{e^{-2t}}{t^2} \]

\[ y_c(t) = c_1 e^{-2t} + c_2 te^{-2t} \]
\[ y(t) = c_1 e^{-2t} + c_2 te^{-2t} - e^{-2t} \ln |t| \]
Solution (1 of 2)

Let \( y_1(t) = e^{-2t} \) and \( y_2(t) = te^{-2t} \), then

\[
W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1 - 2t)e^{-2t} \end{vmatrix} = e^{-4t}.
\]
Solution (1 of 2)

Let $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$, then

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1 - 2t)e^{-2t} \end{vmatrix} = \frac{1}{t}$$

$$\mu_1'(t) = \frac{-y_2(t)g(t)}{W(y_1, y_2)(t)} = \frac{-te^{-2t}(e^{-2t}/t^2)}{e^{-4t}} = -\frac{1}{t}$$

$$\mu_1(t) = -\ln |t|$$
Solution (1 of 2)

Let \( y_1(t) = e^{-2t} \) and \( y_2(t) = te^{-2t} \), then

\[
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\[
\mu'_1(t) = \frac{-y_2(t)g(t)}{W(y_1, y_2)(t)} = \frac{-te^{-2t}(e^{-2t}/t^2)}{e^{-4t}} = \frac{-1}{t}
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\mu_1(t) = -\ln|t|
\]

\[
\mu'_2(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} = \frac{e^{-2t}(e^{-2t}/t^2)}{e^{-4t}} = \frac{1}{t^2}
\]

\[
\mu_2(t) = -\frac{1}{t}
\]
Solution (2 of 2)

The particular solution is

\[ Y(t) = \mu_1(t)y_1(t) + \mu_2(t)y_2(t) \]
\[ = -\ln |t| e^{-2t} - \frac{1}{t} te^{-2t} \]
\[ = -\ln |t| e^{-2t} - e^{-2t} \]

Since the second term is a complementary solution, we may ignore it.

Thus the general solution to the nonhomogeneous equation is

\[ y(t) = (c_1 + c_2 t)e^{-2t} - \ln |t| e^{-2t}. \]
Homework

- Read Section 3.7
- Exercises: 1–19 odd