Linear Independence and the Wronskian
MATH 365 Ordinary Differential Equations

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Let functions $p(t)$ and $q(t)$ be continuous functions on the open interval $I = (\alpha, \beta)$.

For any function $\phi(t)$ which is twice differentiable on $I$, we define the **linear operator** $L$ as follows.

$$L[\phi] = \phi'' + p \phi' + q \phi$$
Operator Notation

- Let functions $p(t)$ and $q(t)$ be continuous functions on the open interval $I = (\alpha, \beta)$.
- For any function $\phi(t)$ which is twice differentiable on $I$, we define the linear operator $L$ as follows.
  \[ L[\phi] = \phi'' + p \phi' + q \phi \]
- $L[\phi]$ is a function defined on $I$. We can evaluate this function as
  \[ L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t) \]
  for all $t \in I$. 
Existence and Uniqueness of Solutions

Theorem
Consider the initial value problem

\[ L[y] = y'' + p(t)y' + q(t)y = g(t) \]
\[ y(t_0) = y_0 \]
\[ y'(t_0) = y'_0 \]

where \( p, q, \) and \( g \) are continuous on an open interval \( I \) that contains the point \( t_0 \). Then there is exactly one solution \( y = \phi(t) \) to this problem and the solution is defined throughout the interval \( I \).
Example

Find the longest interval in which the solution to the initial value problem

\[(t^2 - 3t)y'' + yy' - (t + 3)y = 0\]

\[y(1) = 2\]

\[y'(1) = 1\]

is certain to exist.
Principle of Superposition

Theorem

If $y_1$ and $y_2$ are two solutions to the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

then the linear combination $y = c_1 y_1 + c_2 y_2$ is also a solution for any values of the constants $c_1$ and $c_2$. 
Consider the IVP:

\[ L[y] = y'' + p(t)y' + q(t)y = 0 \]
\[ y(t_0) = y_0 \]
\[ y'(t_0) = y'_0. \]

Suppose \( y_1(t) \) and \( y_2(t) \) solve the ODE. Can a solution to the IVP be written as a linear combination of \( y_1 \) and \( y_2 \)?
Solution

Since \( y_1 \) and \( y_2 \) solve the ODE, then by the Principle of Superposition, \( y = c_1 y_1 + c_2 y_2 \) also solve the ODE. We need merely pick \( c_1 \) and \( c_2 \) so that the initial conditions are satisfied.

\[
c_1 y_1(t_0) + c_2 y_2(t_0) = y_0
\]

\[
c_1 y'_1(t_0) + c_2 y'_2(t_0) = y_0
\]
Solution

Since $y_1$ and $y_2$ solve the ODE, then by the Principle of Superposition, $y = c_1 y_1 + c_2 y_2$ also solve the ODE. We need merely pick $c_1$ and $c_2$ so that the initial conditions are satisfied.

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$
$$c_1 y'_1(t_0) + c_2 y'_2(t_0) = y_0$$

The solution to this linear system of equations can be found using Cramer’s Rule.
Cramer's Rule

\[ c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \]
\[ c_1 y'_1(t_0) + c_2 y'_2(t_0) = y_0 \]

Has solution

\[
c_1 = \frac{y_0 y'_2(t_0) - y'_0 y_2(t_0)}{y_1(t_0)y'_2(t_0) - y'_1(t_0)y_2(t_0)} = \frac{\begin{vmatrix} y_0 & y'_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}}
\]

\[
c_2 = \frac{-y_0 y'_1(t_0) + y'_0 y_1(t_0)}{y_1(t_0)y'_2(t_0) - y'_1(t_0)y_2(t_0)} = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}}.
\]

Provided the denominator determinant is nonzero.
Wronskian Determinant

The expression

\[ W(y_1, y_2)(t_0) = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \]

is called the **Wronskian determinant**.
Important Result

Theorem
Suppose that $y_1$ and $y_2$ are two solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

and the initial conditions

$$y(t_0) = y_0$$
$$y'(t_0) = y'_0$$

are assigned. Then it is always possible to choose constants $c_1$ and $c_2$ so that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the differential equation and initial conditions if and only if $W(y_1, y_2)(t_0) \neq 0$. 
Theorem

Suppose that $y_1$ and $y_2$ are two solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$ 

Then the family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary coefficients $c_1$ and $c_2$ includes every solution to the ODE if and only if there is a point $t_0$ where the Wronskian of $y_1$ and $y_2$ is nonzero.
General Solution

Theorem
Suppose that $y_1$ and $y_2$ are two solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$ 

Then the family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary coefficients $c_1$ and $c_2$ includes every solution to the ODE if and only if there is a point $t_0$ where the Wronskian of $y_1$ and $y_2$ is nonzero.

Remarks:
- We call $y = c_1 y_1(t) + c_2 y_2(t)$ with arbitrary coefficients the general solution.
- The set $S = \{y_1(t), y_2(t)\}$ is called a fundamental set of solutions if and only if their Wronskian is nonzero.
Definition
Two functions $f$ and $g$ are said to be \textbf{linearly dependent} on an interval $I$ if there exist two constants $k_1$ and $k_2$, not both zero, such that

$$k_1 f(t) + k_2 g(t) = 0$$

for all $t \in I$. If the equation above holds for all $t \in I$ only if $k_1 = k_2 = 0$ then $f$ and $g$ are said to be \textbf{linearly independent}. 
Examples

Determine if the following pairs of functions are linearly dependent or independent on $\mathbb{R}$.

$\begin{align*}
\text{f}(t) &= e^{3t} \text{ and } g(t) = e^{3(t-1)} \\
\text{f}(t) &= t^2 + t + 1 \text{ and } g(t) = 3t^2 - 2t + 1
\end{align*}$
Wronskian

The Wronskian can be used to determine if two functions are linearly independent on an interval.

Theorem
If $f$ and $g$ are differentiable functions on an open interval $I$, and $W(f, g)(t_0) \neq 0$ for some $t_0 \in I$, then $f$ and $g$ are linearly independent on $I$. Moreover, if $f$ and $g$ are linearly dependent on $I$, then $W(f, g)(t) = 0$ for all $t \in I$.

Proof.
Suppose $W(f, g)(t_0) \neq 0$ and suppose $c_1 f(t) + c_2 g(t)$ is zero in $I$, then $c_1 f(t_0) + c_2 g(t_0) = 0$.

$c_1 f'(t_0) + c_2 g'(t_0) = 0$
Wronskian

The **Wronskian** can be used to determine if two functions are linearly independent on an interval.

**Theorem**

If \( f \) and \( g \) are differentiable functions on an open interval \( I \) and if \( W(f, g)(t_0) \neq 0 \) for some \( t_0 \in I \), then \( f \) and \( g \) are linearly independent on \( I \). Moreover, if \( f \) and \( g \) are linearly dependent on \( I \), then \( W(f, g)(t) = 0 \) for all \( t \in I \).
Wronskian

The **Wronskian** can be used to determine if two functions are linearly independent on an interval.

**Theorem**

If $f$ and $g$ are differentiable functions on an open interval $I$, and if $W(f, g)(t_0) \neq 0$ for some $t_0 \in I$, then $f$ and $g$ are linearly independent on $I$. Moreover, if $f$ and $g$ are linearly dependent on $I$, then $W(f, g)(t) = 0$ for all $t \in I$.

**Proof.**

Suppose $W(f, g)(t_0) \neq 0$ and suppose $c_1 f(t) + c_2 g(t)$ is zero in $I$, then

\[
\begin{align*}
  c_1 f(t_0) + c_2 g(t_0) &= 0 \\
  c_1 f'(t_0) + c_2 g'(t_0) &= 0
\end{align*}
\]
Examples

Use the Wronskian to establish the linear dependence or independence of the following pairs of functions on $\mathbb{R}$.

- $f(t) = e^{4t}$ and $g(t) = e^{-4t}$
- $f(t) = 2t^3$ and $g(t) = -3t^3$
- $f(t) = 1$ and $g(t) = \cos t$
Theorem (Abel’s Theorem)

If \( y_1 \) and \( y_2 \) are solutions to the ODE

\[
L[y] = y'' + p(t)y' + q(t)y = 0
\]

where \( p \) and \( q \) are continuous on an open interval \( I \), then the Wronskian \( W(y_1, y_2)(t) \) is given by

\[
W(y_1, y_2)(t) = ce^{-\int p(t)\,dt}
\]

where \( c \) is a constant that depends on \( y_1 \) and \( y_2 \), but not on \( t \). Either \( W(y_1, y_2)(t) = 0 \) for all \( t \in I \) (because \( c = 0 \)) or \( W(y_1, y_2)(t) \neq 0 \) for all \( t \in I \) (because \( c \neq 0 \)).
Proof.
By assumption

\[ y_1'' + p(t)y_1' + q(t)y_1 = 0 \]
\[ y_2'' + p(t)y_2' + q(t)y_2 = 0. \]

Multiplying the first equation by \(-y_2\) and multiplying the second equation by \(y_1\) yields

\[ -y_1''y_2 - p(t)y_1'y_2 - q(t)y_1y_2 = 0 \]
\[ y_1y_2'' + p(t)y_1y_2' + q(t)y_1y_2 = 0. \]
Abel’s Theorem (3 of 3)

Proof.

\[-y_1''y_2 - p(t)y_1'y_2 - q(t)y_1y_2 = 0\]
\[y_1y_2'' + p(t)y_1'y_2 + q(t)y_1y_2 = 0\]

Adding the two equations produces

\[y_1y_2''' - y_1''y_2 + p(t) (y_1y_2' - y_1'y_2) = 0\]
\[\frac{dW}{dt} + p(t)W = 0\]
Use Abel’s Theorem to find the general form of the Wronskian for the following second order linear homogeneous ODEs.

1. \( t^2 y'' + ty' + (t^2 - 1)y = 0 \)
2. \( (1 - t^2)y'' - 2ty' + 2y = 0 \)
The following theorem follows from Abel’s Theorem:

**Theorem**

Let $y_1$ and $y_2$ be two solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where $p$ and $q$ are continuous on an open interval $I$. Then $y_1$ and $y_2$ are linearly dependent on $I$ if and only if $W(y_1, y_2)(t) = 0$ for all $t \in I$. 
Homework

▶ Read Section 3.2
▶ Exercises: 1–39 odd