Bisection Method
MATH 375 Numerical Analysis

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We begin to study a set of **root-finding** techniques, starting with the simplest, the **Bisection Method**.
The Bisection Method approximates the root of an equation on an interval by repeatedly halving the interval.

The Bisection Method operates under the conditions necessary for the **Intermediate Value Theorem** to hold.

Suppose \( f \in C[a, b] \) and \( f(a)f(b) < 0 \), then there exists \( p \in (a, b) \) such that \( f(p) = 0 \).

Remark: The root \( p \) found is not necessarily unique.
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**Remark:** The root \( p \) found is not necessarily unique.
Given the continuous function $f(x)$ on the interval $[a, b]$ where $f(a) f(b) < 0$:

**INPUT** endpoints $a$, $b$, tolerance $\epsilon$, maximum iterations $N$.

**STEP 1** Set $i = 1$; $FA = f(a)$.

**STEP 2** While $i \leq N$ do Steps 3–6.

**STEP 3** Set $p = a + \frac{b - a}{2}$; $FP = f(p)$.

**STEP 4** If $FP = 0$ or $\frac{b - a}{2} < \epsilon$ then OUTPUT $p$; STOP.

**STEP 5** Set $i = i + 1$.

**STEP 6** If $FA \cdot FP > 0$ then set $a = p$;

$FA = FP$, else $b = p$.

**STEP 7** OUTPUT “Method failed after $N$ iterations.”; STOP.
The Bisection Method generates a sequence \( \{p_n\}_{n=1}^N \).

We used a stopping criterion of

- \( f(p_n) = 0 \) (in case we hit the root “exactly”), or
- \( \frac{b - a}{2} < \epsilon \) (the original interval is halved enough times that the distance between \( p_{n-1} \) and \( p_n \) is smaller than a specified tolerance), or
- \( i > N \) (the maximum number of iterations is reached).
Alternative Stopping Criteria

Other logic for halting the algorithm includes:

- \(|p_n - p_{n-1}| < \epsilon\)
- \(|f(p_n)| < \epsilon\)
- \(|p_n - p_{n-1}| < \epsilon\) provided \(p_n \neq 0\)
- \(\frac{|p_n - p_{n-1}|}{|p_n|} < \epsilon\)
- \(\frac{|p_n - p_{n-1}|}{\min\{|a_n|, |b_n|\}} < \epsilon\)

**Remark:** the stopping criterion chosen will depend on the equation being solved. There is no “best” criterion.
Approximate the root of \( f(x) = e^x - 4x \) on \([0, 1]\) with \( \epsilon = 10^{-2} \) and \( N = 10 \).
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Remark: \( p_5 \approx 0.357403 \) and hence \( p_5 \) is a better approximation than \( p_6 \).
Approximate the root of \( f(x) = e^x - 4x \) on \([0, 1]\) with \( \epsilon = 10^{-2} \) and \( N = 10 \).

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Theorem

If $f \in C[a, b]$ and $f(a)f(b) < 0$, the Bisection Method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a root $p$ of $f$ with the property that

$$|p_n - p| \leq \frac{b - a}{2^n}, \quad \text{for } n \geq 1.$$
Proof.

If \( n = 1 \) then \( p_1 = \frac{a + b}{2} \) and \( p \in (a, b). \)
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- By induction on $n$ then $|p_n - p| \leq \frac{b - a}{2^n}$.
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1. If $n = 1$ then $p_1 = \frac{a + b}{2}$ and $p \in (a, b)$.

2. Thus $|p_1 - p| \leq \frac{b - a}{2}$.

3. By induction on $n$ then $|p_n - p| \leq \frac{b - a}{2^n}$.

$$p_n = p + O\left(\frac{1}{2^n}\right)$$
Determine the minimum number of iterations of the Bisection Method necessary to approximate a root of \( f(x) = e^x - 4x \) on \([0, 1]\) with \( \epsilon = 10^{-4} \).
Example

Determine the minimum number of iterations of the Bisection Method necessary to approximate a root of $f(x) = e^x - 4x$ on $[0, 1]$ with $\epsilon = 10^{-4}$.

$$|p_n - p| \leq \frac{b - a}{2^n}$$

$$\frac{1 - 0}{2^n} \leq 10^{-4}$$

$$2^n \geq 10^4$$

$$n \geq 14$$
Homework

- Read Section 2.1.
- Exercises: 3a, 11, 13, 15, 16, 17