Composite Numerical Integration
MATH 375

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Outline

1. Introduction
2. Composite Simpson’s Rule
3. Composite Trapezoidal Rule
4. Composite Midpoint Rule
Throughout this presentation we will use the following definite integral to evaluate the performance of the various approximations.

\[
\int_{0}^{2\pi} e^{3x} \sin 2x \, dx = \frac{1}{13} e^{3\pi} (3 \sin 2x - 2 \cos 2x) \bigg|_{0}^{2\pi} \\
= \frac{2}{13} (1 - e^{6\pi}) \\
\approx -2.36235 \times 10^{7}
\]
Composite Simpson’s Rule

Theorem

Let \( f \in C^4[a, b] \), \( n \in \mathbb{N} \) be even, \( h = (b - a)/n \), and \( x_j = a + jh \) for \( j = 0, 1, \ldots, n \). There exists \( \mu \in (a, b) \) for which the Composite Simpson’s Rule for \( n \) subintervals can be written with its error term as

\[
\int_a^b f(x) \, dx = \frac{h}{3} \left[ f(a) + 2 \sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4 \sum_{j=1}^{n/2} f(x_{2j-1}) + f(b) \right] - \frac{b - a}{180} h^4 f^{(4)}(\mu).
\]
Composite Simpson’s Rule with \( n = 128 \).

**Example**

\[
\int_{0}^{2\pi} e^{3x} \sin 2x \, dx \approx -2.36233 \times 10^7
\]

Absolute error \( \approx 227.661 \)

Error bound \( \approx 3734.45 \)
**Theorem**

Let \( f \in C^2[a, b] \), \( n \in \mathbb{N} \), \( h = (b - a)/n \), and \( x_j = a + jh \) for \( j = 0, 1, \ldots, n \). There exists \( \mu \in (a, b) \) for which the **Composite Trapezoidal Rule** for \( n \) subintervals can be written with its error term as

\[
\int_a^b f(x) \, dx = \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] - \frac{b - a}{12} h^2 f''(\mu).
\]
Composite Trapezoidal Rule with \( n = 1024 \).

**Example**

\[
\int_{0}^{2\pi} e^{3x} \sin 2x \, dx \approx -2.36226 \times 10^7
\]

Absolute error \( \approx 963.519 \)

Error bound \( \approx 36324.3 \)
Theorem

Let \( f \in C^2[a, b] \), \( n \in \mathbb{N} \) be even, \( h = (b - a)/(n + 2) \), and \( x_j = a + (j + 1)h \) for \( j = -1, 0, \ldots, n + 1 \). There exists \( \mu \in (a, b) \) for which the Composite Midpoint Rule for \( n + 2 \) subintervals can be written with its error term as

\[
\int_a^b f(x) \, dx = 2h \sum_{j=0}^{n/2} f(x_{2j}) + \frac{b - a}{6} h^2 f''(\mu).
\]
Composite Midpoint Rule with $n = 4096$.

Example

\[
\int_{0}^{2\pi} e^{3x} \sin 2x \, dx \approx -2.36236 \times 10^7
\]

Absolute error $\approx 120.323$

Error bound $\approx 4536.11$