Linear Systems of Equations
MATH 375

J. Robert Buchanan

Department of Mathematics

Fall 2013
Linear systems are equations of the form:

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    \vdots \\
    a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{align*}
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a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{align*}

- The coefficients \( a_{ij} \) for \( i, j = 1, 2, \ldots, n \) are constants.
- The expressions \( b_i \) for \( i = 1, 2, \ldots, n \) are constant.
- The unknowns are \( x_i \) for \( i = 1, 2, \ldots, n \).
We will study **direct methods** for solving such systems of linear equations.

Direct methods can solve a linear system in a **predictable**, **fixed** number of steps.

We will employ the following operations to solve linear systems:

- An equation can be replaced by a nonzero multiple of itself.
- An equation can be replaced by the sum of itself and another equation.
- Any two equations can be swapped.
Use of these elementary operations will enable us to convert a linear system into an equivalent linear system in **reduced** or **triangular** form. Then **back-substitution** can be used to solve the equivalent system.

**Example**

Derive the equivalent linear system in triangular form.

\[
\begin{align*}
2x_1 + 4x_2 - x_3 &= -5 \\
x_1 + x_2 - 3x_3 &= -9 \\
4x_1 + x_2 + 2x_3 &= 9
\end{align*}
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x_1 + x_2 - 3x_3 &= -9 \quad \text{(mult. 1st by } \frac{1}{2}, \text{ subt.)} \\
4x_1 + x_2 + 2x_3 &= 9 \quad \text{(mult. 1st by 2, subt.)} 
\end{align*} \]
Solution (1 of 2)

\[
\begin{align*}
2x_1 + 4x_2 - x_3 & = -5 \\
    x_1 + x_2 - 3x_3 & = -9 \quad \text{(mult. 1st by } \frac{1}{2}, \text{ subt.)} \\
4x_1 + x_2 + 2x_3 & = 9 \quad \text{(mult. 1st by 2, subt.)}
\end{align*}
\]

\[
\begin{align*}
2x_1 + 4x_2 - x_3 & = -5 \\
-x_2 - \frac{5}{2}x_3 & = -\frac{13}{2} \\
-7x_2 + 4x_3 & = 19 \quad \text{(mult. 2nd by 7, subt.)}
\end{align*}
\]
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\[\begin{align*}
2x_1 + 4x_2 - x_3 &= -5 \\
-x_2 - \frac{5}{2}x_3 &= -\frac{13}{2} \quad \text{(mult. by } -1) \\
\frac{43}{2}x_3 &= \frac{129}{2}
\end{align*}\]
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\[ 2x_1 + 4x_2 - x_3 = -5 \]
\[ 2x_2 + 5x_3 = 13 \]
\[ 43x_3 = 129 \]
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x_2 + \frac{5}{2}x_3 &= \frac{13}{2} \quad \text{(mult. by 2)} \\
\frac{43}{2}x_3 &= \frac{129}{2} \quad \text{(mult. by 2)}
\end{align*}\]

The final linear system is in triangular (reduced) form.
Solving for \((x_1, x_2, x_3)\)

\[
\begin{align*}
2x_1 + 4x_2 - x_3 &= -5 \\
2x_2 + 5x_3 &= 13 \\
43x_3 &= 129
\end{align*}
\]

1. We may solve the 3rd equation for \(x_3\) and substitute this result into the 1st and 2nd equations.
2. We may solve the 2nd equation for \(x_2\) and substitute this result into the 1st equation.
3. We may solve the 1st equation for \(x_1\).
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1. We may solve the 3rd equation for \(x_3\) and substitute this result into the 1st and 2nd equations.
2. We may solve the 2nd equation for \(x_2\) and substitute this result into the 1st equation.
3. We may solve the 1st equation for \(x_1\).

\[
x_3 = \frac{129}{43} = 3
\]

\[
2x_2 + 5(3) = 13 \quad \implies \quad x_2 = -1
\]

\[
2x_1 + 4(-1) - 3 = -5 \quad \implies \quad x_1 = 1
\]
Matrix Notation

Since we only manipulated the constants and coefficients of the linear system, we only need keep track of them and can suppress the unknowns.
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**Notation:**

\[
A = [a_{ij}] = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1m} \\
    a_{21} & a_{22} & \cdots & a_{2m} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nm}
\end{bmatrix}
\]

is called an \( n \times m \) **matrix**.
A $1 \times n$ matrix

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n}
\end{bmatrix}
\]

is called a **row vector**.

An $n \times 1$ matrix

\[
\begin{bmatrix}
a_{11} \\
a_{21} \\
\vdots \\
a_{n1}
\end{bmatrix}
\]

is called a **column vector**.
Vectors

A $1 \times n$ matrix

\[
\begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n}
\end{bmatrix}
\]

is called a **row vector**.

An $n \times 1$ matrix

\[
\begin{bmatrix}
    a_{11} \\
    a_{21} \\
    \vdots \\
    a_{n1}
\end{bmatrix}
\]

is called a **column vector**.

**Remark:** we can represent any linear system with an $n \times n$ matrix and an $n \times 1$ column vector.
The linear system of the form:

\[ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \]
\[ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \]
\[ \vdots \]
\[ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \]

will often be represented by an \( n \times (n + 1) \) augmented matrix:

\[
\begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
    a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \\
\end{bmatrix}
= \tilde{A}.
\]
Reducing the $A$ portion of an augmented matrix to triangular form and then using back substitution to solve the linear system is called **Gaussian elimination with back substitution.**
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Assuming that for some $i \in \{1, 2, \ldots, n\}$ we have $a_{i1} \neq 0$ the reduced form of $\tilde{A}$ resembles the following.

\[
\tilde{A} = \begin{bmatrix}
\tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} & \tilde{b}_1 \\
0 & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} & \tilde{b}_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \tilde{a}_{nn} & \tilde{b}_n
\end{bmatrix}
\]
Back Substitution

Given

\[
\begin{bmatrix}
\tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} & \tilde{b}_1 \\
0 & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} & \tilde{b}_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \tilde{a}_{nn} & \tilde{b}_n \\
\end{bmatrix},
\]

we have the solution:

\[
\begin{align*}
x_n &= \frac{\tilde{b}_n}{\tilde{a}_{nn}} \\
\vdots \\
x_i &= \frac{\tilde{b}_i - \tilde{a}_{in}x_n - \tilde{a}_{i,n-1}x_{n-1} - \cdots - \tilde{a}_{i,i+1}x_{i+1}}{\tilde{a}_{ii}} = \frac{\tilde{b}_i - \sum_{j=i+1}^n \tilde{a}_{ij}x_j}{\tilde{a}_{ii}} \\
\vdots \\
x_1 &= \frac{\tilde{b}_1 - \tilde{a}_{1n}x_n - \tilde{a}_{1,n-1}x_{n-1} - \cdots - \tilde{a}_{12}x_2}{\tilde{a}_{11}} = \frac{\tilde{b}_1 - \sum_{j=2}^n \tilde{a}_{1j}x_j}{\tilde{a}_{11}}
\end{align*}
\]
Comment: at each stage of the reduction process we have assumed that $\tilde{a}_{ii} \neq 0$. If we encounter $\tilde{a}_{ii} = 0$ then we look for $j > i$ (a lower row) such that $\tilde{a}_{ji} \neq 0$ and swap rows $i$ and $j$. 

Use Gaussian elimination with back substitution to solve the following linear system given in augmented matrix form.

\[
\begin{bmatrix}
1 & -1 & 3 \\
2 & 3 & -1 \\
1 & 1 & 0 \end{bmatrix}
\begin{bmatrix}
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Linear Systems of Equations
\end{bmatrix}
\]
Comment: at each stage of the reduction process we have assumed that \( \tilde{a}_{ii} \neq 0 \). If we encounter \( \tilde{a}_{ii} = 0 \) then we look for \( j > i \) (a lower row) such that \( \tilde{a}_{ji} \neq 0 \) and swap rows \( i \) and \( j \).

This operation is called pivoting.
Pivoting

Comment: at each stage of the reduction process we have assumed that $\tilde{a}_{ii} \neq 0$. If we encounter $\tilde{a}_{ij} = 0$ then we look for $j > i$ (a lower row) such that $\tilde{a}_{ji} \neq 0$ and swap rows $i$ and $j$.

This operation is called \textbf{pivoting}.

Use Gaussian elimination with back substitution to solve the following linear system given in augmented matrix form.

$$
\begin{bmatrix}
1 & -1 & 3 & 2 \\
3 & -3 & 1 & -1 \\
1 & 1 & 0 & 3
\end{bmatrix}
$$
Solution

Gaussian elimination:

\[
\begin{bmatrix}
1 & -1 & 3 & 2 \\
3 & -3 & 1 & -1 \\
1 & 1 & 0 & 3
\end{bmatrix} \mapsto \begin{bmatrix}
1 & -1 & 3 & 2 \\
0 & 0 & -8 & -7 \\
0 & 2 & -3 & 1
\end{bmatrix} \mapsto \begin{bmatrix}
1 & -1 & 3 & 2 \\
0 & 2 & -3 & 1 \\
0 & 0 & -8 & -7
\end{bmatrix}
\]

Back substitution:

\[
x_3 = \frac{7}{8}
\]
\[
x_2 = \frac{1 - (-3)(7/8)}{2} = \frac{29}{16}
\]
\[
x_1 = \frac{2 - 3(7/8) - (-1)(29/16)}{1} = \frac{19}{16}
\]
**Question:** how many multiplications/divisions and how many additions/subtractions are necessary to reduce the augmented matrix?
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Suppose that at the beginning of the $i$th stage of the reduction the augmented matrix resembles the following.

$$
\tilde{A} = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} & a_{1,n+1} \\
  0 & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} & a_{2,n+1} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & a_{ii} & \cdots & a_{in} & a_{i,n+1} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & a_{ni} & \cdots & a_{nn} & a_{n,n+1}
\end{bmatrix}
$$
To row reduce below the $i$th row we must

- Find row multipliers requiring multiplications/divisions.
- Multiply the $i$th row by each of the row multipliers requiring multiplications/divisions.
- Add a multiple of the $i$th row to the rows beneath requiring additions/subtractions.

Summary: to row reduce below the $i$th row requires $(n-i) + (n-i)(n-i+1) = (n-i)(n-i+2)$ multiplications/divisions and $(n-i)(n-i+1)$ additions/subtractions.
To row reduce below the $i$th row we must

- Find $n - i$ row multipliers requiring $n - i$ multiplications/divisions.
- Multiply the $i$th row by each of the $n - i$ row multipliers requiring multiplications/divisions.
- Add a multiple of the $i$th row to the $n - i$ rows beneath requiring additions/subtractions.

Summary: to row reduce below the $i$th row requires $(n - i) + (n - i)(n - i + 1) = (n - i)(n - i + 2)$ multiplications/divisions and $(n - i)(n - i + 1)$ additions/subtractions.
To row reduce below the $i$th row we must

- Find $n - i$ row multipliers requiring $n - i$ multiplications/divisions.
- Multiply the $i$th row by each of the $n - i$ row multipliers requiring $(n - i)(n - i + 1)$ multiplications/divisions.
- Add a multiple of the $i$th row to the $n - i$ rows beneath requiring additions/subtractions.

Summary: to row reduce below the $i$th row requires $(n - i) + (n - i)(n - i + 1) = (n - i)(n - i + 2)$ multiplications/divisions and $(n - i)(n - i + 1)$ additions/subtractions.

Starting from the original matrix, this set of row reductions must be carried out $n - 1$ times.
To row reduce below the $i$th row we must

- Find $n - i$ row multipliers requiring $n - i$ multiplications/divisions.
- Multiply the $i$th row by each of the $n - i$ row multipliers requiring $(n - i)(n - i + 1)$ multiplications/divisions.
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To row reduce below the $i$th row we must

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- Add a multiple of the $i$th row to the $n - i$ rows beneath requiring $(n - i)(n - i + 1)$ additions/subtractions.

**Summary:** to row reduce below the $i$th row requires

$$(n-i) + (n-i)(n-i+1) = (n-i)(n-i+2)$$ multiplication/divisions

and $(n - i)(n - i + 1)$ additions/subtractions.
To row reduce below the $i$th row we must

- Find $n - i$ row multipliers requiring $n - i$ multiplications/divisions.
- Multiply the $i$th row by each of the $n - i$ row multipliers requiring $(n - i)(n - i + 1)$ multiplications/divisions.
- Add a multiple of the $i$th row to the $n - i$ rows beneath requiring $(n - i)(n - i + 1)$ additions/subtractions.

**Summary:** to row reduce below the $i$th row requires

$$(n - i) + (n - i)(n - i + 1) = (n - i)(n - i + 2)$$ multiplications/divisions

and $(n - i)(n - i + 1)$ additions/subtractions.

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To row reduce below the $i$th row we must

- Find $n - i$ row multipliers requiring $n - i$ multiplications/divisions.
- Multiply the $i$th row by each of the $n - i$ row multipliers requiring $(n - i)(n - i + 1)$ multiplications/divisions.
- Add a multiple of the $i$th row to the $n - i$ rows beneath requiring $(n - i)(n - i + 1)$ additions/subtractions.

**Summary:** to row reduce below the $i$th row requires

$$(n-i)+(n-i)(n-i+1) = (n-i)(n-i+2)$$ multiplications/divisions

and $(n - i)(n - i + 1)$ additions/subtractions.

Starting from the original matrix, this set of row reductions must be carried out $n - 1$ times.
Total multiplications/divisions for matrix reduction:

\[ \sum_{i=1}^{n-1} (n - i)(n - i + 2) \]

\[ = \sum_{i=1}^{n-1} (n^2 + 2n) - (2n + 2) \sum_{i=1}^{n-1} i + \sum_{i=1}^{n-1} i^2 \]

\[ = (n^2 + 2n)(n - 1) - (2n + 2) \frac{n(n - 1)}{2} + \frac{(n - 1)(n)(2n - 1)}{6} \]

\[ = \frac{2n^3 + 3n^2 - 5n}{6} \]
Total additions/subtractions for matrix reduction:

\[\sum_{i=1}^{n-1} (n - i)(n - i + 1)\]

\[= \sum_{i=1}^{n-1} (n^2 + n) - (2n + 1) \sum_{i=1}^{n-1} i + \sum_{i=1}^{n-1} i^2\]

\[= (n^2 + n)(n - 1) - (2n + 1) \frac{n(n - 1)}{2} + \frac{(n - 1)(n)(2n - 1)}{6}\]

\[= \frac{n^3 - n}{3}\]
Growth of operation counts required for row reduction:

<table>
<thead>
<tr>
<th>n</th>
<th>Ops</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0 \times 10^7</td>
<td>200</td>
</tr>
<tr>
<td>1.0 \times 10^8</td>
<td>400</td>
</tr>
<tr>
<td>1.5 \times 10^8</td>
<td>600</td>
</tr>
<tr>
<td>2.0 \times 10^8</td>
<td>800</td>
</tr>
<tr>
<td>2.5 \times 10^8</td>
<td>1000</td>
</tr>
<tr>
<td>3.0 \times 10^8</td>
<td></td>
</tr>
</tbody>
</table>
**Question:** how many multiplications/divisions and how many additions/subtractions are necessary to perform the back substitution necessary to solve for $x_i$?
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\[
x_i = \frac{\tilde{b}_i - \tilde{a}_{in}x_n - \tilde{a}_{i,n-1}x_{n-1} - \cdots - \tilde{a}_{i,i+1}x_{i+1}}{\tilde{a}_{ii}}
\]
**Question:** how many multiplications/divisions and how many additions/subtractions are necessary to perform the back substitution necessary to solve for $x_i$?

$$x_i = \frac{\tilde{b}_i - \tilde{a}_{in}x_n - \tilde{a}_{i,n-1}x_{n-1} - \cdots - \tilde{a}_{i,i+1}x_{i+1}}{\tilde{a}_{ii}}$$

- **Multiplications/divisions:** $n - i + 1$ (if $i \neq n$) or 1 (if $i = n$)
- **Additions/subtractions:** $n - i$
**Question:** how many multiplications/divisions and how many additions/subtractions are necessary to perform the back substitution necessary to solve for $x_i$?

$$x_i = \frac{\tilde{b}_i - \tilde{a}_{in}x_n - \tilde{a}_{i,n-1}x_{n-1} - \cdots - \tilde{a}_{i,i+1}x_{i+1}}{\tilde{a}_{ii}}$$

Multiplications/divisions: $n - i + 1$ (if $i \neq n$) or $1$ (if $i = n$)

Additions/subtractions: $n - i$

This operation must be carried out $n - 1$ times.
Total multiplications/divisions for back substitution:

\[ 1 + \sum_{i=1}^{n-1} (n - i + 1) = 1 + \sum_{i=1}^{n-1} (n + 1) - \sum_{i=1}^{n-1} i \]

\[ = 1 + (n - 1)(n + 1) - \frac{n(n - 1)}{2} \]

\[ = \frac{n^2 + n}{2} \]

Total additions/subtractions for back substitution:

\[ \sum_{i=1}^{n-1} (n - i) = \sum_{i=1}^{n-1} n - \sum_{i=1}^{n-1} i \]

\[ = n(n - 1) - \frac{n(n - 1)}{2} \]

\[ = \frac{n^2 - n}{2} \]
Thus the total operation counts for solving a linear system via Gaussian elimination and back substitution are

multiplications/divisions: \( \frac{n^3 + 3n^2 - n}{3} \)

additions/subtractions: \( \frac{2n^3 + 3n^2 - 5n}{6} \)
Algorithm: Gaussian Elimination with Back Substitution

Given the augmented matrix $A = [a_{ij}]_{i=1,\ldots,n, j=1,\ldots,n+1}$:

**STEP 1** For $i = 1, 2, \ldots, n - 1$ set

$$p = \min_{j \in \{i, i+1, \ldots, n\}} \{ j \mid a_{ji} \neq 0 \}$$

- If $p \neq i$ then transpose rows $i$ and $p$.
- For $j = i + 1, i + 2, \ldots, n$ replace row $j$ by the sum of row $j$ and $-\frac{a_{ji}}{a_{ii}}$ times row $i$.

**STEP 2** Set $x_n = \frac{a_{n,n+1}}{a_{nn}}$.

**STEP 3** For $i = n - 1, n - 2, \ldots, 1$ set

$$x_i = \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}$$
If no value for $p$ (the pivot) can be found, then no unique solution to the linear system exists.

If $a_{nn} = 0$ then no unique solution exists.
Example

Solve the following linear system using Gaussian elimination with back substitution and 3-digit chopping arithmetic.

\[
\begin{align*}
3.33x_1 + 15900x_2 - 10.3x_3 &= 15900 \\
2.22x_1 + 16.7x_2 + 9.61x_3 &= 28.5 \\
1.56x_1 + 5.17x_2 + 16.8x_3 &= 8.42
\end{align*}
\]
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\end{align*}
\]

For comparison purposes, the exact solution is nearly:

\[
\begin{align*}
x_1 &= 7.5073 \\
x_2 &= 0.998102 \\
x_3 &= -0.50307
\end{align*}
\]
Augmented matrix:

\[
\begin{bmatrix}
3.33 & 15900 & -10.3 & 15900 \\
2.22 & 16.7 & 9.61 & 28.5 \\
1.56 & 5.17 & 16.8 & 8.42
\end{bmatrix}
\]
Augmented matrix:

\[
\begin{bmatrix}
3.33 & 15900 & -10.3 & 15900 \\
2.22 & 16.7 & 9.61 & 28.5 \\
1.56 & 5.17 & 16.8 & 8.42 \\
\end{bmatrix}
\]

Multiply row 1 by 0.666 and subtract from row 2.

\[
\begin{bmatrix}
2.21 & 10500 & -6.85 & 10500 \\
\end{bmatrix}
\]
Augmented matrix:

$$
\begin{bmatrix}
3.33 & 15900 & -10.3 & 15900 \\
2.22 & 16.7 & 9.61 & 28.5 \\
1.56 & 5.17 & 16.8 & 8.42
\end{bmatrix}
$$

Multiply row 1 by 0.666 and subtract from row 2.

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\begin{bmatrix}
2.21 & 10500 & -6.85 & 10500 \\
3.33 & 15900 & -10.3 & 15900 \\
0.01 & -10500 & 16.4 & -10500 \\
1.56 & 5.17 & 16.8 & 8.42
\end{bmatrix}
$$
Augmented matrix:

\[
\begin{bmatrix}
3.33 & 15900 & -10.3 & 15900 \\
0.01 & -10500 & 16.4 & -10500 \\
1.56 & 5.17 & 16.8 & 8.42
\end{bmatrix}
\]
Augmented matrix:

\[
\begin{bmatrix}
3.33 & 15900 & -10.3 & 15900 \\
0.01 & -10500 & 16.4 & -10500 \\
1.56 & 5.17 & 16.8 & 8.42
\end{bmatrix}
\]

Multiply row 1 by 0.468 and subtract from row 3.

\[
\begin{bmatrix}
1.55 & 7440 & -4.82 & 7440
\end{bmatrix}
\]
Augmented matrix:

\[
\begin{bmatrix}
3.33 & 15900 & -10.3 & 15900 \\
0.01 & -10500 & 16.4 & -10500 \\
1.56 & 5.17 & 16.8 & 8.42 \\
\end{bmatrix}
\]

Multiply row 1 by 0.468 and subtract from row 3.

\[
\begin{bmatrix}
1.55 & 7440 & -4.82 & 7440 \\
3.33 & 15900 & -10.3 & 15900 \\
0.01 & -10500 & 16.4 & -10500 \\
0.01 & -7440 & 21.6 & -7440 \\
\end{bmatrix}
\]
Solution (3 of 4)

Partially reduced matrix:

\[
\begin{bmatrix}
3.33 & 15900 & -10.3 & 15900 \\
0.01 & -10500 & 16.4 & -10500 \\
0.01 & -7440 & 21.6 & -7440
\end{bmatrix}
\]

Multiply row 2 by 0.708 and subtract from row 3.
Partially reduced matrix:

\[
\begin{bmatrix}
3.33 & 15900 & -10.3 & 15900 \\
0.01 & -10500 & 16.4 & -10500 \\
0.01 & -7440 & 21.6 & -7440 \\
\end{bmatrix}
\]

Multiply row 2 by 0.708 and subtract from row 3.

\[
\begin{bmatrix}
0.01 & -7430 & 11.6 & -7430 \\
\end{bmatrix}
\]
Partially reduced matrix:

\[
\begin{bmatrix}
3.33 & 15900 & -10.3 & 15900 \\
0.01 & -10500 & 16.4 & -10500 \\
0.01 & -7440 & 21.6 & -7440 \\
\end{bmatrix}
\]

Multiply row 2 by 0.708 and subtract from row 3.

\[
\begin{bmatrix}
0.01 & -7430 & 11.6 & -7430 \\
\end{bmatrix}
\]
Given the reduced matrix:

\[
\begin{bmatrix}

3.33 & 15900 & -10.3 & | & 15900 \\
0.01 & -10500 & 16.4 & | & -10500 \\
0.01 & -10.0 & 10.0 & | & -10.0
\end{bmatrix}
\]

Begin back substitution.

This does not compare very well with the "exact" solution.
Given the reduced matrix:

\[
\begin{bmatrix}
3.33 & 15900 & -10.3 & 15900 \\
0.01 & -10500 & 16.4 & -10500 \\
0.01 & -10.0 & 10.0 & -10.0 \\
\end{bmatrix}
\]

Begin back substitution.

\[
x_3 = -1.00 \\
x_2 = \frac{-10500 - 16.4(-1.00)}{-10500} = 1.00 \\
x_1 = \frac{15900 - (-10.3)(-1.00) - (15900)(1.00)}{3.33} = 0.00
\]
Given the reduced matrix:

\[
\begin{bmatrix}
3.33 & 15900 & -10.3 & 15900 \\
0.01 & -10500 & 16.4 & -10500 \\
0.01 & -10.0 & 10.0 & -10.0
\end{bmatrix}
\]

Begin back substitution.

\[
x_3 = -1.00
\]
\[
x_2 = \frac{-10500 - 16.4(-1.00)}{-10500} = 1.00
\]
\[
x_1 = \frac{15900 - (-10.3)(-1.00) - (15900)(1.00)}{3.33} = 0.00
\]

This does not compare very well with the “exact” solution.

\[
x_3 = -0.50307
\]
\[
x_2 = 0.998102
\]
\[
x_1 = 7.5073
\]
 Homework

- Read Section 6.1.
- Exercises: 1ad, 3, 5ad, 9, 12, 15