We have learned how to approximate a function using Lagrange polynomials and how to estimate the error in such an approximation.

Today we will learn how to interpolate between data values found in a table without knowing the function that generated the values.

We will see that we can perform the interpolation without explicitly writing out the interpolating polynomial.
Suppose we had only the following table of data for $f(x)$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.1</td>
<td>0.17537</td>
</tr>
<tr>
<td>22.2</td>
<td>0.37784</td>
</tr>
<tr>
<td>32.0</td>
<td>0.52992</td>
</tr>
<tr>
<td>41.6</td>
<td>0.66393</td>
</tr>
<tr>
<td>50.5</td>
<td>0.63608</td>
</tr>
</tbody>
</table>

Approximate $f(27.5)$ using polynomial interpolation.
Suppose we had only the following table of data for \( f(x) \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.1</td>
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<td>41.6</td>
<td>0.66393</td>
</tr>
<tr>
<td>50.5</td>
<td>0.63608</td>
</tr>
</tbody>
</table>

Approximate \( f(27.5) \) using polynomial interpolation. Linear approximation:

\[
P_1(27.5) = \frac{(27.5 - 32.0)}{(22.2 - 32.0)} f(22.2) + \frac{(27.5 - 22.2)}{(32.0 - 22.2)} f(32.0) \approx 0.46009
\]
Higher Order Approximations

Quadratic approximation: (two choices)

\[ P_2(27.5) = \frac{(27.5 - 22.2)(27.5 - 32.0)}{(10.1 - 22.2)(10.1 - 32.0)} f(10.1) \]
\[ + \frac{(27.5 - 10.1)(27.5 - 32.0)}{(22.2 - 10.1)(22.2 - 32.0)} f(22.2) \]
\[ + \frac{(27.5 - 10.1)(27.5 - 22.2)}{(32.0 - 10.1)(32.0 - 22.2)} f(32.0) \approx 0.46141 \]

\[ \hat{P}_2(27.5) = \frac{(27.5 - 32.0)(27.5 - 41.6)}{(22.2 - 32.0)(22.2 - 41.6)} f(22.2) \]
\[ + \frac{(27.5 - 22.2)(27.5 - 41.6)}{(32.0 - 22.2)(32.0 - 41.6)} f(32.0) \]
\[ + \frac{(27.5 - 22.2)(27.5 - 32.0)}{(41.6 - 22.2)(41.6 - 32.0)} f(41.6) \approx 0.46200 \]

There are also two potential cubic interpolating polynomials and a single quartic polynomial.
Without knowing $f(x)$ we have no idea of the size of the errors in these approximations.

The highest degree polynomial does not necessarily deliver the smallest error.

Knowing the Lagrange Interpolating Polynomial of degree $k$ does not help us find the one of degree $k + 1$. 

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J. Robert Buchanan
Neville’s Method
Remarks

- Without knowing $f(x)$ we have no idea of the size of the errors in these approximations.
- The highest degree polynomial does not necessarily deliver the smallest error.
- Knowing the Lagrange Interpolating Polynomial of degree $k$ does not help us find the one of degree $k + 1$.
- There is a method for using previous calculations to derive higher order interpolations.
Definition

If $f$ is a function defined at $x_0, x_1, \ldots, x_n$ distinct real numbers and $m_1, m_2, \ldots, m_k$ are $k$ distinct integers with $0 \leq m_i \leq n$, then the Lagrange polynomial that agrees with $f$ at the $k$ points $x_{m_1}, x_{m_2}, \ldots, x_{m_k}$ is denoted $P_{m_1,m_2,\ldots,m_k}$. 

J. Robert Buchanan

Neville's Method
Let \( x_0 = 1, \ x_1 = 2, \ x_2 = 4, \ x_3 = 5, \) and \( x_4 = 7 \) and \( f(x) = J_0(x) \) (\textit{Bessel function} of the first kind of order zero).

Determine \( P_{1,3,4}(x) \).
Let $x_0 = 1$, $x_1 = 2$, $x_2 = 4$, $x_3 = 5$, and $x_4 = 7$ and $f(x) = J_0(x)$ (Bessel function of the first kind of order zero).

Determine $P_{1,3,4}(x)$.

\[
P_{1,3,4}(x) = \frac{(x - 5)(x - 7)}{(2 - 5)(2 - 7)} J_0(2) + \frac{(x - 2)(x - 7)}{(5 - 2)(5 - 7)} J_0(5) \\
+ \frac{(x - 2)(x - 5)}{(7 - 2)(7 - 5)} J_0(7) \\
= \left[ \frac{1}{15} x^2 - \frac{4}{5} x + \frac{7}{3} \right] J_0(2) + \left[ -\frac{1}{6} x^2 + \frac{3}{2} x + \frac{7}{3} \right] J_0(5) \\
+ \left[ \frac{1}{10} x^2 - \frac{7}{10} x + 1 \right] J_0(7)
\]
Let $x_0 = 1$, $x_1 = 2$, $x_2 = 4$, $x_3 = 5$, and $x_4 = 7$ and $f(x) = J_0(x)$.

Use $P_{1,3,4}(3)$ to approximate $J_0(3)$. 

$$P_{1,3,4}(3) = \frac{(x - 5)(x - 7)}{(2 - 5)(2 - 7)} J_0(2) + \frac{(x - 2)(x - 7)}{(5 - 2)(5 - 7)} J_0(5) + \frac{(x - 2)(x - 5)}{(7 - 2)(7 - 5)} J_0(7) \approx -0.0590053$$

Actual value: $J_0(3) \approx -0.260052$
Let $x_0 = 1$, $x_1 = 2$, $x_2 = 4$, $x_3 = 5$, and $x_4 = 7$ and $f(x) = J_0(x)$.

Use $P_{1,3,4}(3)$ to approximate $J_0(3)$.

\[ P_{1,3,4}(x) = \frac{(x - 5)(x - 7)}{(2 - 5)(2 - 7)} J_0(2) + \frac{(x - 2)(x - 7)}{(5 - 2)(5 - 7)} J_0(5) \]

\[ + \frac{(x - 2)(x - 5)}{(7 - 2)(7 - 5)} J_0(7) \]

\[ P_{1,3,4}(3) = \frac{8}{15} J_0(2) + \frac{2}{3} J_0(5) - \frac{1}{5} J_0(7) \]

\[ \approx -0.0590053 \]
Let $x_0 = 1$, $x_1 = 2$, $x_2 = 4$, $x_3 = 5$, and $x_4 = 7$ and $f(x) = J_0(x)$.

Use $P_{1,3,4}(3)$ to approximate $J_0(3)$.

\[
P_{1,3,4}(x) = \frac{(x - 5)(x - 7)}{(2 - 5)(2 - 7)} J_0(2) + \frac{(x - 2)(x - 7)}{(5 - 2)(5 - 7)} J_0(5) + \frac{(x - 2)(x - 5)}{(7 - 2)(7 - 5)} J_0(7)
\]

\[
P_{1,3,4}(3) = \frac{8}{15} J_0(2) + \frac{2}{3} J_0(5) - \frac{1}{5} J_0(7)
\]

\[
\approx -0.0590053
\]

Actual value: $J_0(3) \approx -0.260052$
Theoretical Result

Theorem

Let $f$ be defined at $x_0, x_1, \ldots, x_k$ and $x_i$ and $x_j$ be two distinct real numbers in this set. Then the $k$th degree Lagrange polynomial that interpolates $f$ at $x_0, x_1, \ldots, x_k$ is

$$P(x) = \frac{(x - x_j)}{(x_i - x_j)} P_{0,1,...,j-1,j+1,...,k}(x) - \frac{(x - x_i)}{(x_i - x_j)} P_{0,1,...,i-1,i+1,...,k}(x).$$
Theoretical Result

Theorem

Let $f$ be defined at $x_0, x_1, \ldots, x_k$ and $x_i$ and $x_j$ be two distinct real numbers in this set. Then the $k$th degree Lagrange polynomial that interpolates $f$ at $x_0, x_1, \ldots, x_k$ is

$$P(x) = \frac{(x - x_j)}{(x_i - x_j)} P_{0,1,...,i-1,i+1,...,k}(x) - \frac{(x - x_i)}{(x_i - x_j)} P_{0,1,...,i-1,i+1,...,k}(x).$$

This theorem allows us to build a Lagrange polynomial of degree $k$ from two Lagrange polynomials each of degree $k - 1$. 
Proof

Let $Q(x) = P_{0,1,...,j-1,j+1,...,k}(x)$ and $\hat{Q}(x) = P_{0,1,...,i-1,i+1,...,k}(x)$, then

$$P(x) = \frac{(x - x_j)Q(x) - (x - x_i)\hat{Q}(x)}{(x_i - x_j)}.$$ 

Thus $P(x)$ is a polynomial of degree at most $k+1$ which interpolates $f(x)$ at $x_i$ for $i = 0, 1, \ldots, k$ and thus $P(x) = P_{0,1,...,k}(x)$. 

J. Robert Buchanan

Neville's Method
Proof

Let \( Q(x) = P_{0,1,...,j-1,j+1,...,k}(x) \) and \( \hat{Q}(x) = P_{0,1,...,i-1,i+1,...,k}(x) \), then

\[
P(x) = \frac{(x - x_j)Q(x) - (x - x_i)\hat{Q}(x)}{(x_i - x_j)}.
\]

If \( x = x_i \), then \( P(x_i) = Q(x_i) = f(x_i) \). If \( x = x_j \), then \( P(x_j) = \hat{Q}(x_j) = f(x_j) \). If \( l \in \{0, 1, \ldots, k\} \) with \( i \neq l \neq j \) then

\[
P(x_l) = \frac{(x_l - x_j)Q(x_l) - (x_l - x_i)\hat{Q}(x_l)}{(x_i - x_j)} = \frac{(x_l - x_j)f(x_l) - (x_l - x_i)f(x_l)}{(x_i - x_j)} = f(x_l).
\]
Proof

Let \( Q(x) = P_{0,1,...,j-1,j+1,...,k}(x) \) and \( \hat{Q}(x) = P_{0,1,...,i-1,i+1,...,k}(x) \), then

\[
P(x) = \frac{(x - x_j)Q(x) - (x - x_i)\hat{Q}(x)}{(x_i - x_j)}.
\]

If \( x = x_i \), then \( P(x_i) = Q(x_i) = f(x_i) \). If \( x = x_j \), then \( P(x_j) = \hat{Q}(x_j) = f(x_j) \). If \( l \in \{0, 1, \ldots, k\} \) with \( i \neq l \neq j \) then

\[
P(x_l) = \frac{(x_l - x_j)Q(x_l) - (x_l - x_i)\hat{Q}(x_l)}{(x_i - x_j)} = \frac{(x_l - x_j)f(x_l) - (x_l - x_i)f(x_l)}{(x_i - x_j)} = f(x_l).
\]

Thus \( P(x) \) is a polynomial of degree at most \( k + 1 \) which interpolates \( f(x) \) at \( x_i \) for \( i = 0, 1, \ldots, k \) and thus

\[
P(x) = P_{0,1,...,k}(x).
\]
Recursive Generation of Polynomials

\[ P_{0,1}(x) = \frac{1}{x_1 - x_0} \left[ (x - x_0)P_1 - (x - x_1)P_0 \right] \]

\[ P_{1,2}(x) = \frac{1}{x_2 - x_1} \left[ (x - x_1)P_2 - (x - x_2)P_1 \right] \]

\[ P_{0,1,2}(x) = \frac{1}{x_2 - x_0} \left[ (x - x_0)P_{1,2} - (x - x_2)P_{0,1} \right] \]

\vdots
Define polynomial $Q_{i,j}(x) = P_{i-j,i-j+1,...,i}(x)$ where $0 \leq j \leq i$.

- $j$ represents the degree of the polynomial.
- $i$ determines the consecutive values of $x$ used.
- $Q_{i,j}(x)$ can be found recursively.

$$Q_{i,j}(x) = \frac{(x - x_{i-j})Q_{i,j-1}(x) - (x - x_i)Q_{i-1,j-1}(x)}{x_i - x_{i-j}}$$
Given \( x_0, x_1, \ldots, x_n \) and \( f(x_0), f(x_1), \ldots f(x_n) \) we can arrange the Lagrange interpolating polynomials in the following table.

\[
\begin{array}{c|c|c}
 x_0 & f(x_0) = P_0 = Q_{0,0} & \\
 x_1 & f(x_1) = P_1 = Q_{1,0} & P_{0,1} = Q_{1,1} \\
 x_2 & f(x_2) = P_2 = Q_{2,0} & P_{1,2} = Q_{2,1} & P_{0,1,2} = Q_{2,2} \\
 \vdots & \vdots & \vdots & \vdots \\
 x_n & f(x_n) = P_n = Q_{n,0} & P_{n-1,n} = Q_{n,1} & P_{n-2,n-1,n} = Q_{n,2} & \cdots & P_{0,1,\ldots,n} = Q_{n,n}
\end{array}
\]
Example

Approximate $f(27.5)$ from the following data.

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$f(x_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.1</td>
<td>0.17537</td>
</tr>
<tr>
<td>22.2</td>
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<td>0.66393</td>
</tr>
<tr>
<td>50.5</td>
<td>0.63608</td>
</tr>
</tbody>
</table>
Order points by increasing distance from $x = 27.5$.

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$Q_{i,0}$</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
<td>22.2</td>
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Example

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$Q_{i,0}$</th>
<th>$Q_{i,1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>32.0</td>
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<td></td>
</tr>
<tr>
<td>22.2</td>
<td>0.37784</td>
<td>0.46009†</td>
</tr>
<tr>
<td>41.6</td>
<td>0.66393</td>
<td>0.45600‡</td>
</tr>
<tr>
<td>10.1</td>
<td>0.17537</td>
<td>0.44524*</td>
</tr>
<tr>
<td>50.5</td>
<td>0.63608</td>
<td>0.37380**</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{†}0.46009 &= \frac{(27.5 - 32.0)(0.37784) - (27.5 - 22.2)(0.52992)}{22.2 - 32.0} \\
\text{‡}0.45600 &= \frac{(27.5 - 22.2)(0.66393) - (27.5 - 41.6)(0.37784)}{41.6 - 22.2} \\
\text{*}0.44524 &= \frac{(27.5 - 41.6)(0.17537) - (27.5 - 10.1)(0.66393)}{10.1 - 41.6} \\
\text{**}0.37380 &= \frac{(27.5 - 10.1)(0.63608) - (27.5 - 50.5)(0.17537)}{50.5 - 10.1}
\end{align*}
\]
### Example

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$Q_{i,0}$</th>
<th>$Q_{i,1}$</th>
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</tr>
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<tbody>
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</tr>
<tr>
<td>41.6</td>
<td>0.66393</td>
<td>0.45600</td>
<td>0.46200†</td>
</tr>
<tr>
<td>10.1</td>
<td>0.17537</td>
<td>0.44524</td>
<td>0.46071‡</td>
</tr>
<tr>
<td>50.5</td>
<td>0.63608</td>
<td>0.37380</td>
<td>0.55843*</td>
</tr>
</tbody>
</table>

†0.46200 = \(\frac{(27.5 - 32.0)(0.45600) - (27.5 - 41.6)(0.46009)}{41.6 - 32.0}\)

‡0.46071 = \(\frac{(27.5 - 22.2)(0.44524) - (27.5 - 10.1)(0.45600)}{10.1 - 22.2}\)

*0.55843 = \(\frac{(27.5 - 41.6)(0.37380) - (27.5 - 50.5)(0.44524)}{50.5 - 41.6}\)
### Example

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$Q_{i,0}$</th>
<th>$Q_{i,1}$</th>
<th>$Q_{i,2}$</th>
<th>$Q_{i,3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>32.0</td>
<td>0.52992</td>
<td></td>
<td></td>
<td></td>
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<td>0.66393</td>
<td>0.45600</td>
<td>0.46200</td>
<td></td>
</tr>
<tr>
<td>10.1</td>
<td>0.17537</td>
<td>0.44524</td>
<td>0.46071</td>
<td>0.46174†</td>
</tr>
<tr>
<td>50.5</td>
<td>0.63608</td>
<td>0.37380</td>
<td>0.55843</td>
<td>0.47901‡</td>
</tr>
</tbody>
</table>

†0.46174 = \( \frac{(27.5 - 32.0)(0.46071) - (27.5 - 10.1)(0.46200)}{10.1 - 32.0} \)

‡0.47901 = \( \frac{(27.5 - 22.2)(0.55843) - (27.5 - 50.5)(0.46071)}{50.5 - 22.2} \)
Example

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$Q_{i,0}$</th>
<th>$Q_{i,1}$</th>
<th>$Q_{i,2}$</th>
<th>$Q_{i,3}$</th>
<th>$Q_{i,4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>32.0</td>
<td>0.52992</td>
<td></td>
<td></td>
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<td>22.2</td>
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<td>0.46174</td>
<td></td>
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<td>50.5</td>
<td>0.63608</td>
<td>0.37380</td>
<td>0.55843</td>
<td>0.47901</td>
<td>0.45754†</td>
</tr>
</tbody>
</table>

$\hat{Q}_{4,5} = \frac{(27.5 - 32.0)(0.47901) - (27.5 - 50.5)(0.46174)}{50.5 - 32.0}$

$\hat{Q}_{4,5} = 0.45754$
**Goal:** evaluate the interpolating polynomial $P$ on $n + 1$ distinct numbers $x_0, \ldots, x_n$ at $x$ to approximate $f(x)$.

**INPUT** values $x, x_0, \ldots, x_n, f(x_0), \ldots, f(x_n)$

**STEP 1** For $i = 0, 1, \ldots, n$ set $Q_{i,0} = f(x_i)$.

**STEP 2** For $i = 1, 2, \ldots, n$
  
  for $j = 1, 2, \ldots, i$
  
  set $Q_{i,j} = \frac{1}{x_i - x_j} \left[ (x - x_{i-j})Q_{i,j-1} - (x - x_i)Q_{i-1,j-1} \right]$.

**STEP 3** OUTPUT $Q$. STOP.
Homework

- Read Section 3.2.
- Exercises: 1a, 3, 5, 7