Richardson’s Extrapolation
MATH 375 Numerical Analysis

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Department of Mathematics

Fall 2013
Recall the centered-difference formula for $f'(x_0)$:

$$f'(x_0) = \frac{1}{2h} [f(x_0 + h) - f(x_0 - h)] - \frac{h^2}{6} f^{(3)}(z)$$

In today’s lesson we will learn to create higher-accuracy approximations while using lower-order formulas.

The technique, known as **extrapolation** can be used whenever the truncation error has a predictable form (as above) and depends on a parameter such as $h$, the step size.
Suppose that $N_1(h)$ is a formula which approximates a quantity $M$.

$$N_1(h) \approx M$$
General Setting

Suppose that $N_1(h)$ is a formula which approximates a quantity $M$.

$$N_1(h) \approx M$$

Imagine the truncation error of this approximation can be written as

$$M - N_1(h) = K_1 h + K_2 h^2 + K_3 h^3 + \cdots$$

for some unknown constants $K_1, K_2, K_3, \ldots$. 

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Richardson's Extrapolation
Suppose that \( N_1(h) \) is a formula which approximates a quantity \( M \).

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N_1(h) \approx M
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\[
M - N_1(h) = K_1 h + K_2 h^2 + K_3 h^3 + \cdots
\]

for some unknown constants \( K_1, K_2, K_3, \ldots \).

Note that

\[
M - N_1(0.1) = K_1(0.1) + K_2(0.1)^2 + \cdots \approx (0.1)K_1
\]

\[
M - N_1(0.01) = K_1(0.01) + K_2(0.01)^2 + \cdots \approx (0.01)K_1
\]

and in general \( M - N_1(h) \approx K_1 h \).
Question: Since the truncation error is $O(h)$, can we combine several $O(h)$ approximations to create an $O(h^n)$ approximation where $n \geq 2$?
**Question:** Since the truncation error is $O(h)$, can we combine several $O(h)$ approximations to create an $O(h^n)$ approximation where $n \geq 2$?

\[
M = N_1(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots
\]

\[
M = N_1 \left( \frac{h}{2} \right) + K_1 \frac{h}{2} + K_2 \frac{h^2}{4} + K_3 \frac{h^3}{8} + \cdots
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\]

Multiply the 2nd equation by 2 and subtract the 1st equation.

\[
M = 2N_1 \left( \frac{h}{2} \right) - N_1(h) + K_2 \left[ \frac{h^2}{2} - h^2 \right] + K_3 \left[ \frac{h^3}{4} - h^3 \right] + \cdots \\
= N_1 \left( \frac{h}{2} \right) + \left[ N_1 \left( \frac{h}{2} \right) - N_1(h) \right] - \frac{K_2}{2} h^2 - \frac{3K_3}{4} h^3 - \cdots 
\]

Note: the $O(h)$ truncation terms have vanished.
Define $N_2(h) = N_1 \left( \frac{h}{2} \right) + \left[ N_1 \left( \frac{h}{2} \right) - N_1(h) \right]$ and then we have

$$M = N_2(h) - \frac{K_2}{2} h^2 - \frac{3K_3}{4} h^3 - \ldots$$

which has $O(h^2)$ truncation error.

**Note:** we have combined multiple $O(h)$ approximations to generate an $O(h^2)$ approximation.
Recall the 2-point forward-difference formula for \( f'(x_0) \):

\[
f'(x_0) = \frac{1}{h}(f(x_0 + h) - f(x_0)) - \frac{f''(z)}{2} h
\]

This is an \( O(h) \) truncation error.
Recall the 2-point forward-difference formula for $f'(x_0)$:

$$f'(x_0) = \frac{1}{h}(f(x_0 + h) - f(x_0)) - \frac{f''(z)}{2}h$$

This is an $O(h)$ truncation error.

Let $f(x) = x \sin x$ then we have:

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Applying the extrapolation formula gives us another approximation:

$$N_2(0.1) = N_1(0.05) + (N_1(0.05) - N_1(0.1)) = 1.38436$$
Example

Recall the 2-point forward-difference formula for $f'(x_0)$:

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$$N_2(0.1) = N_1(0.05) + (N_1(0.05) - N_1(0.1)) = 1.38436$$

Note that $|N_2(0.1) - f'(1)| \approx 0.00259168$. 

J. Robert Buchanan  Richardson's Extrapolation
Example

Use Richardson’s Extrapolation and the 2-point forward difference formula for \( f'(x_0) \) to develop an \( O(h^2) \) approximation to \( f'(2) \) where \( f(x) = x^2 \cos x \) using \( h = 0.1 \).
Use Richardson’s Extrapolation and the 2-point forward difference formula for $f'(x_0)$ to develop an $O(h^2)$ approximation to $f'(2)$ where $f(x) = x^2 \cos x$ using $h = 0.1$.

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$$N_2(0.1) = N_1(0.05) + (N_1(0.05) - N_1(0.1)) = -5.30499$$

Note that $|N_2(0.1) - f'(2)| \approx 0.00320877$. 
Remark: If the truncation error contains only even powers of $h$, the extrapolation is more effective.

Suppose

\[ M = N_1(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots \]

\[ M = N_1 \left( \frac{h}{2} \right) + K_1 \frac{h^2}{4} + K_2 \frac{h^4}{16} + K_3 \frac{h^6}{64} + \cdots \]
**Remark:** If the truncation error contains only even powers of $h$, the extrapolation is more effective.

Suppose

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\]

Multiply the 2nd equation by 4 and subtract the 1st equation.

\[
3M = \left[ 4N_1 \left( \frac{h}{2} \right) - N_1(h) \right] + K_2 \left[ \frac{h^4}{4} - h^4 \right] + K_3 \left[ \frac{h^6}{16} - h^6 \right] + \cdots
\]
If we multiply the previous equation by $1/3$ we obtain

$$M = \frac{1}{3} \left[ 4N_1 \left(\frac{h}{2}\right) - N_1(h) \right] + \frac{K_2}{3} \left[ \frac{h^4}{4} - h^4 \right] + \frac{K_3}{3} \left[ \frac{h^6}{16} - h^6 \right] + \cdots$$
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Define

$$N_2(h) = \frac{1}{3} \left[ 4N_1 \left( \frac{h}{2} \right) - N_1(h) \right] = N_1 \left( \frac{h}{2} \right) + \frac{1}{3} \left[ N_1 \left( \frac{h}{2} \right) - N_1(h) \right].$$
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\]

This is an approximation formula with truncation error \(O(h^4)\).

\[
M = N_2(h) - \frac{K_2}{4} h^4 - \frac{5K_3}{16} h^6 + \cdots
\]
Example

Recall the 3-point centered-difference formula for $f'(x_0)$:

$$f'(x_0) = \frac{1}{2h} (f(x_0 + h) - f(x_0 - h)) - \frac{f'''(z)}{6} h^2$$

This is an $O(h^2)$ truncation error.

Let $f(x) = x \sin x$ then we have:

\[
\begin{align*}
&h N_1(1) \approx \text{Abs. Err.} \approx 0.1010 \times 10^{-7} \\
&h N_2(0.1) = N_1(0.05) + \frac{1}{3}(N_1(0.05) - N_1(0.1)) = 1.38177
\end{align*}
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Note that $|N_2(0.1) - f'(1)| \approx 9.88697 \times 10^{-7}$. 

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Use Richardson’s Extrapolation and the 3-point centered-difference formula for $f'(x_0)$ to develop an $O(h^4)$ approximation to $f'(2)$ where $f(x) = x^2 \cos x$ using $h = 0.1$. 

Applying the extrapolation formula gives us another approximation:

$$N_2(0.1) = N_1(0.05) + \frac{3}{1} (N_1(0.05) - N_1(0.1)) = -5.30178$$

Note that $|N_2(0.1) - f'(2)| \approx 1.29563 \times 10^{-6}$. 

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Note that \( |N_2(0.1) - f'(2)| \approx 1.29563 \times 10^{-6} \).
Return to the $O(h^4)$ Formula

Recall:

$$M = N_2(h) - \frac{K_2}{4} h^4 - \frac{5K_3}{16} h^6 + \cdots$$
Return to the $O(h^4)$ Formula

Recall:

\[ M = N_2(h) - \frac{K_2}{4} h^4 - \frac{5K_3}{16} h^6 + \cdots \]

Replace $h$ by $h/2$:

\[ M = N_2\left(\frac{h}{2}\right) - \frac{K_2}{64} h^4 - \frac{5K_3}{1024} h^6 + \cdots \]

which is also has an $O(h^4)$ truncation error.
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which is also has an \( O(h^4) \) truncation error.

Multiply the 2nd equation by 16 and subtract the first equation from it.

\[ 15M = \left[ 16N_2 \left( \frac{h}{2} \right) - N_2(h) \right] + \frac{15K_3}{64} h^6 + \cdots \]
Multiplying both sides of the last equation by $1/15$ yields:

\[ M = \frac{1}{15} \left[ 16N_2 \left( \frac{h}{2} \right) - N_2(h) \right] + \frac{K_3}{64} h^6 + \cdots \]
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We can define

$$N_3(h) = N_2 \left( \frac{h}{2} \right) + \frac{1}{15} \left[ N_2 \left( \frac{h}{2} \right) - N_2(h) \right].$$

This approximation formula has an $O(h^6)$ truncation error.
We will use two approximations to $f'(1)$ where $f(x) = x \sin x$ with $O(h^4)$ truncation errors to develop an approximation with $O(h^6)$ truncation error.
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Note that \( |N_3(0.1) - f'(1)| \approx 1.99358 \times 10^{-11} \).
Use Richardson’s Extrapolation and the 3-point centered-difference formula for $f'(x_0)$ to develop an $O(h^6)$ approximation to $f'(2)$ where $f(x) = x^2 \cos x$ using $h = 0.1$. 

Applying the extrapolation formula gives us another approximation:

$$N_3(0.1) = N_3(0.05) + 1.15 \left( N_2(0.05) - N_2(0.1) \right) = -5.30178 \times 10^{-6}$$

Note that $|N_3(0.1) - f'(2)| \approx 7.09512 \times 10^{-11}$. 

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Note that \( |N_3(0.1) - f'(2)| \approx 7.09512 \times 10^{-11} \).
For $j = 2, 3, \ldots$ the $O(h^{2j})$ truncation error approximation is given by the formula

$$N_j(h) = N_{j-1} \left( \frac{h}{2} \right) + \frac{1}{4^{j-1} - 1} \left[ N_{j-1} \left( \frac{h}{2} \right) - N_{j-1}(h) \right].$$
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For $j = 2, 3, \ldots$ the $O(h^j)$ truncation error approximation is given by the formula

$$N_j(h) = N_{j-1} \left( \frac{h}{2} \right) + \frac{1}{2^{j-1} - 1} \left[ N_{j-1} \left( \frac{h}{2} \right) - N_{j-1}(h) \right].$$
**Remark:** Richardson’s extrapolation provides a convenient means for developing the 5-point approximations to $f'(x_0)$.

Assuming $f \in C^5[a, b]$ and $x_0 \in (a, b)$ expand $f(x)$ as a degree 4 Taylor polynomial about $x_0$.

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 \\
+ \frac{1}{6}f'''(x_0)(x - x_0)^3 + \frac{1}{24}f^{(4)}(x_0)(x - x_0)^4 \\
+ \frac{1}{120}f^{(5)}(z)(x - x_0)^5
\]

where $z$ lies between $x$ and $x_0$. 
Evaluate the Taylor polynomial expansion at $x = x_0 \pm h$.

\[
\begin{align*}
\quad \quad f(x_0 + h) &= f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \frac{1}{6}f'''(x_0)h^3 \\
&\quad + \frac{1}{24}f^{(4)}(x_0)h^4 + \frac{1}{120}f^{(5)}(z_1)h^5 \\
\quad \quad f(x_0 - h) &= f(x_0) - f'(x_0)h + \frac{1}{2}f''(x_0)h^2 - \frac{1}{6}f'''(x_0)h^3 \\
&\quad + \frac{1}{24}f^{(4)}(x_0)h^4 - \frac{1}{120}f^{(5)}(z_2)h^5
\end{align*}
\]

with $x_0 - h \leq z_2 \leq x_0 \leq z_1 \leq x_0 + h$. 

Now subtract the 2nd equation from the 1st equation.
Evaluate the Taylor polynomial expansion at $x = x_0 \pm h$.

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \frac{1}{6}f'''(x_0)h^3 + \frac{1}{24}f^{(4)}(x_0)h^4 + \frac{1}{120}f^{(5)}(z_1)h^5$$

$$f(x_0 - h) = f(x_0) - f'(x_0)h + \frac{1}{2}f''(x_0)h^2 - \frac{1}{6}f'''(x_0)h^3 + \frac{1}{24}f^{(4)}(x_0)h^4 - \frac{1}{120}f^{(5)}(z_2)h^5$$

with $x_0 - h \leq z_2 \leq x_0 \leq z_1 \leq x_0 + h$.

Now subtract the 2nd equation from the 1st equation.
\[ f(x_0 + h) - f(x_0 - h) = 2hf'(x_0) + \frac{h^3}{3} f'''(x_0) + \frac{h^5}{120} \left[ f^{(5)}(z_1) + f^{(5)}(z_2) \right] \]
\[ f(x_0 + h) - f(x_0 - h) = 2hf'(x_0) + \frac{h^3}{3} f'''(x_0) + \frac{h^5}{120} \left[ f^{(5)}(z_1) + f^{(5)}(z_2) \right] \]

By assuming that \( f \in \mathcal{C}^5[a, b] \) we know \( f^{(5)}(x) \) is continuous on \([a, b]\).

Note that

\[ \frac{1}{2} \left[ f^{(5)}(z_1) + f^{(5)}(z_2) \right] \]

lies between \( f^{(5)}(z_1) \) and \( f^{(5)}(z_2) \).
Five-Point Formula (2 of 5)

\[ f(x_0+h) - f(x_0-h) = 2hf'(x_0) + \frac{h^3}{3} f'''(x_0) + \frac{h^5}{120} \left[ f^{(5)}(z_1) + f^{(5)}(z_2) \right] \]

By assuming that \( f \in C^5[a, b] \) we know \( f^{(5)}(x) \) is continuous on \([a, b]\).

Note that

\[ \frac{1}{2} \left[ f^{(5)}(z_1) + f^{(5)}(z_2) \right] \]

lies between \( f^{(5)}(z_1) \) and \( f^{(5)}(z_2) \).

According to the Intermediate Value Theorem there exists \( w \) between \( z_1 \) and \( z_2 \) for which

\[ f^{(5)}(w) = \frac{1}{2} \left[ f^{(5)}(z_1) + f^{(5)}(z_2) \right] \]

\[ 2f^{(5)}(w) = f^{(5)}(z_1) + f^{(5)}(z_2) \]
Thus we may write the Taylor polynomial difference as
\[
f(x_0 + h) - f(x_0 - h) = 2hf'(x_0) + \frac{h^3}{3}f'''(x_0) + \frac{h^5}{60}f^{(5)}(w)
\]
for some \(x_0 - h \leq w \leq x_0 + h\).

Solve this equation for \(f'(x_0)\).
Thus we may write the Taylor polynomial difference as

\[
f(x_0 + h) - f(x_0 - h) = 2hf'(x_0) + \frac{h^3}{3}f'''(x_0) + \frac{h^5}{60}f(5)(w)
\]

for some \( x_0 - h \leq w \leq x_0 + h \).

Solve this equation for \( f'(x_0) \).

\[
f'(x_0) = \frac{1}{2h} [f(x_0 + h) - f(x_0 - h)] - \frac{h^2}{6} f'''(x_0) - \frac{h^4}{120} f(5)(w)
\]

Now apply the Richardson’s extrapolation technique to this approximation.
\[ f'(x_0) = \frac{1}{2h} [f(x_0 + h) - f(x_0 - h)] - \frac{h^2}{6} f'''(x_0) - \frac{h^4}{120} f^{(5)}(\tilde{w}) \]

If we replace \( h \) by \( 2h \) we get

\[ f'(x_0) = \frac{1}{4h} [f(x_0 + 2h) - f(x_0 - 2h)] - \frac{4h^2}{6} f'''(x_0) - \frac{16h^4}{120} f^{(5)}(\tilde{\tilde{w}}) \]

where \( \tilde{\tilde{w}} \) lies between \( x_0 - 2h \) and \( x_0 + 2h \).
\[ f'(x_0) = \frac{1}{2h} \left[ f(x_0 + h) - f(x_0 - h) \right] - \frac{h^2}{6} f'''(x_0) - \frac{h^4}{120} f^{(5)}(w) \]

If we replace \( h \) by \( 2h \) we get

\[ f'(x_0) = \frac{1}{4h} \left[ f(x_0 + 2h) - f(x_0 - 2h) \right] - \frac{4h^2}{6} f'''(x_0) - \frac{16h^4}{120} f^{(5)}(\tilde{w}) \]

where \( \tilde{w} \) lies between \( x_0 - 2h \) and \( x_0 + 2h \).

Multiply the 1st equation by 4:

\[ 4f'(x_0) = \frac{4}{2h} \left[ f(x_0 + h) - f(x_0 - h) \right] - \frac{4h^2}{6} f'''(x_0) - \frac{4h^4}{120} f^{(5)}(w) \]

and subtract the 2nd equation.
\[ 3f'(x_0) = \frac{2}{h} [f(x_0 + h) - f(x_0 - h)] - \frac{1}{4h} [f(x_0 + 2h) - f(x_0 - 2h)] \]
\[ \quad - \frac{4h^4}{120} f^{(5)}(w) + \frac{16h^4}{120} f^{(5)}(\tilde{w}) \]

\[ f'(x_0) = \frac{2}{3h} [f(x_0 + h) - f(x_0 - h)] - \frac{1}{12h} [f(x_0 + 2h) - f(x_0 - 2h)] \]
\[ \quad - \frac{h^4}{90} f^{(5)}(w) + \frac{2h^4}{45} f^{(5)}(\tilde{w}) \]
\[ \quad = \frac{1}{12h} [f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)] \]
\[ \quad - \frac{h^4}{30} f^{(5)}(\hat{w}) \]

The form of the truncation error has not been justified.
Read Section 4.2.
Exercises: 1a, 7, 9