Heat Equation in 3D

These notes will briefly outline the derivation of the heat equation in three dimensions. Throughout these notes the following quantities will be referenced.

c: specific heat of material, amount of heat per unit mass necessary to raise the temperature one degree

\( \rho \): density of material, mass per unit volume

\( u(x, y, z, t) \): temperature of the material at location \((x, y, z)\) at time \(t\)

\( Q(x, y, z, t) \): amount of heat energy generated per unit volume per unit time at location \((x, y, z)\) at time \(t\)

\( \phi(x, y, z) \): heat energy flux at location \((x, y, z)\)

\( K_0 \): thermal conductivity of the material, the amount of power per unit length per degree the material can conduct

\( k \): thermal diffusivity of the material

We will also need Gauss’s Theorem (Divergence Theorem):

**Gauss’s Theorem:** Suppose region \( R \subset \mathbb{R}^3 \) is bounded by the closed surface \( S \) and that \( \mathbf{n}(x, y, z) \) denotes the unit outward normal to \( S \). If the components of the vector field \( \mathbf{F}(x, y, z) \) have continuous first partial derivatives in \( R \), then

\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_R \nabla \cdot \mathbf{F} \, dV.
\]

Suppose the material in question occupies the region in three-dimensional space denoted by \( R \) and is bounded by the surface \( S \). Let the unit normal vector to the surface \( S \) be denoted \( \mathbf{n} \). The heat energy contained in \( R \) at time \( t \) is the value of

\[
\iiint_R c \rho u \, dV.
\]

The heat energy leaving region \( R \) through its boundary \( S \) is given by the surface integral

\[
\iint_S \phi \cdot \mathbf{n} \, dS.
\]

The heat energy generated per unit time at time \( t \) in region \( R \) can be written as

\[
\iiint_R Q \, dV.
\]
Thus the conservation law of heat energy in region \( R \) can be expressed as

\[
\frac{d}{dt} \iiint_{R} c \rho u \, dV = - \iiint_{S} \phi \cdot n \, dS + \iiint_{R} Q \, dV
\]

\[
= - \iiint_{R} \nabla \cdot \phi \, dV + \iiint_{R} Q \, dV \quad \text{(Gauss’s Theorem)}
\]

\[
\iiint_{R} c \rho \frac{\partial u}{\partial t} \, dV = \iiint_{R} (Q - \nabla \cdot \phi) \, dV
\]

\[
0 = \iiint_{R} \left( Q - \nabla \cdot \phi - c \rho \frac{\partial u}{\partial t} \right) \, dV
\]

Since \( R \) is an arbitrary three-dimensional region then

\[
c \rho \frac{\partial u}{\partial t} + \nabla \cdot \phi - Q = 0. \quad (1)
\]

In three dimensions Fourier’s Law of Heat Conduction can be stated as

\[
\phi = -K_0 \nabla u.
\]

Substituting this into eq. (1) yields

\[
c \rho \frac{\partial u}{\partial t} + \nabla \cdot (-K_0 \nabla u) - Q = 0
\]

\[
c \rho \frac{\partial u}{\partial t} = K_0 (\nabla \cdot \nabla u) + Q
\]

\[
= K_0 \nabla^2 u + Q
\]

\[
\frac{\partial u}{\partial t} = k \nabla^2 u + \frac{Q}{c \rho}
\]

If there are no sources of heat in region \( R \) then the heat equation simplifies to

\[
\frac{\partial u}{\partial t} = k \nabla^2 u.
\]