Hedging

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Introduction

Definition 1  Hedging is the practice of making a portfolio of investments less sensitive to changes in market variables.

There are various hedging strategies.

For example, Delta hedging attempts to keep the $\Delta$ of a portfolio nearly 0, so that the value of the portfolio is insensitive to changes in the price of a security.
Without a strategy

Example 2  A bank sells $10^6$ European calls on a stock where $S(0) = 50$, $K = 52$, $r = 2.5\%$, $T = \frac{1}{3}$, and $\sigma = 22.5\%$. According to the Black-Scholes option pricing formula, $C = 0.832242$.

Suppose the bank purchases the $10^6$ shares of the security immediately. At $T$ the net revenue satisfies:

$$832242 + \left( \min\{52, S\} e^{-0.025/3} - 50 \right) \cdot 10^6.$$
Suppose the bank purchases the $10^6$ shares of the security at the strike time and immediately sells them to the owner of the option. The net revenue satisfies:

$$832242 + \min\{0, 52 - S\}e^{-0.025/3} \cdot 10^6.$$
Recall: If the value of a solution to the Black-Scholes PDE is \( F \) then \( \Delta = \frac{\partial F}{\partial S} \) where \( S \) is the value of some security underlying \( F \).

- If \( F \) is an option then for every unit change in the value of the underlying security, the value of the option changes by \( \Delta \).

- A portfolio consisting of securities and options is called **Delta-neutral** if for every option sold, \( \Delta \) units of the security are bought.
Example 3  Suppose $S = $100, $r = 4\%$, $\sigma = 23\%$, $K = $105, and $T = 1/4$. Under these conditions $w = -0.279806$, the value of a European call option is $C = 0.731705$ and Delta for the option is

$$\Delta = \frac{\partial C}{\partial S} = 0.389813.$$ 

Thus if a firm sold an investor European call options on ten thousand shares of the security, the firm would receive $7317.05 and purchase $389813 = (10000)(0.389813)(100)$ worth of the security.
Rebalancing a Portfolio

If the firm of the previous example does nothing else until the strike time, we say they have adopted a “hedge and forget” scheme.

The price of the security will (probably) change throughout the life of the option, so the firm may choose to make periodic adjustments to the number shares of the security it holds.

This scheme is known as rebalancing the portfolio.
Extended Example

Assume the value of the security follows the random walk shown below.

The European call option will be exercised since $S(T) > K$. 
End of First Week

- Suppose that \( S(1/52) = 98.79 \).
- Re-compute \( \Delta = 0.339811 \).
- Firm adjusts its security holdings so that it now owns 3398 shares of the security with a total value of $335699.
- Firm has incurred interest costs on the money borrowed to purchase shares of the security. This cost is

\[
(389813)(e^{0.04/52} - 1) = 299.97.
\]
- Cumulative costs to the firm at \( t = 1/52 \) are $335999.
## Weekly Rebalancing

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<th>Week</th>
<th>S</th>
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<th>Shares Held</th>
<th>Interest Cost</th>
<th>Cumulative Cost</th>
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Suppose the call option expires without being exercised.

\[ S(T) < K \]
## Weekly Rebalancing

<table>
<thead>
<tr>
<th>Week</th>
<th>S</th>
<th>Δ</th>
<th>Shares Held</th>
<th>Interest Cost</th>
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Delta Neutral Portfolios

A portfolio consists of a short position in a European call option and a long position in the security (Delta hedged). Thus the net value $\mathcal{P}$ of the portfolio is

$$ \mathcal{P} = C - \Delta S = C - \left. \frac{\partial C}{\partial S} \right|_{S_0} S. $$

$\mathcal{P}$ satisfies the Black-Scholes equation since $C$ and $S$ separately solve it. Thus Delta for the portfolio is

$$ \frac{\partial \mathcal{P}}{\partial S} = \left. \frac{\partial C}{\partial S} - \frac{\partial C}{\partial S} \right|_{S_0}. $$

$$ \frac{\partial \mathcal{P}}{\partial S} \approx 0 \text{ when } S \approx S(0). $$
Taylor Series for $\mathcal{P}$

$$
\mathcal{P} = \mathcal{P}_0 + \frac{\partial \mathcal{P}}{\partial t}(t - t_0) + \frac{\partial \mathcal{P}}{\partial S}(S - S_0) + \frac{\partial^2 \mathcal{P}}{\partial S^2} \frac{(S - S_0)^2}{2} + \cdots
$$

$$
\delta \mathcal{P} = \Theta \delta t + \Delta \delta S + \frac{1}{2} \Gamma (\delta S)^2 + \cdots
$$

$$
\delta \mathcal{P} \approx \Theta \delta t + \frac{1}{2} \Gamma (\delta S)^2
$$

- $\Theta$ is not stochastic and thus must be retained.
- What about $\Gamma$?
Gamma Neutral Portfolios

Recall: \( \Gamma = \frac{\partial^2 F}{\partial S^2} \)

- Since \( \frac{\partial^2}{\partial S^2} (S') = 0 \) a portfolio cannot be made gamma neutral if it contains only an option and its underlying security.
- Portfolio must include an additional component which depends non-linearly on \( S \).
- Portfolio can include two (or more) different types of option dependent on the same security.
Example

Suppose a portfolio contains options with two different strike times written on the same stock.

A firm may sell European call options with a strike time three months and buy European call options on the same stock with a strike time of six months.

Let the number of the early option sold be $w_e$ and the number of the later option be $w_l$. The Gamma of the portfolio would be

$$\Gamma_P = w_e\Gamma_e - w_l\Gamma_l,$$

where $\Gamma_e$ and $\Gamma_l$ denote the Gammas of the earlier and later options respectively.
Example (cont.)

Choose $\Gamma_e$ and $\Gamma_l$ so that $\Gamma_P = 0$.

Introduce the security so as to make the portfolio Delta neutral.

**Question:** What does changing the number of shares of the security in the portfolio affect $\Delta$ but not $\Gamma$?

$$\delta P \approx \Theta \delta t.$$
Example

- Suppose $S = $100, $\sigma = 0.22$, and $r = 2.5\%$.
- An investment firm sells a European call option on this stock with $T_3 = 1/4$ and $K = $102.
- The firm buys European call options on the same stock with the same strike price but with $T_6 = 1/2$.
- Gamma of the 3-month option is $\Gamma_3 = 0.03618$ and Gamma of the 6-month option is $\Gamma_6 = 0.02563$.
- The portfolio is Gamma neutral in the first quadrant of $w_3w_6$-space where the equation

\[ 0.03618w_3 - 0.02563w_6 = 0 \]

is satisfied.
Suppose \( w_3 = 100000 \) of the three-month option were sold.

Portfolio is Gamma neutral if \( w_6 = 141163 \) six-month options are purchased.

Before including the underlying stock in the portfolio, the Delta of the portfolio is

\[
 w_3 \Delta_3 - w_6 \Delta_6 = (100000)(0.4728) - (141163)(0.5123) = -25038.
\]

Portfolio can be made Delta neutral if 25038 shares of the underlying stock are sold short.
Over a wide range of values for the underlying stock, the value of the portfolio remains nearly constant.
Conclusion

- Rho and Vega can be used to hedge portfolios against changes in the interest rate and volatility respectively.
- We have assumed that the necessary options and securities could be bought or sold so as to form the desired hedge.
- If this is not true then a firm or investor may have to substitute a different, but related security or other financial instrument in order to set up the hedge.