Hedging

MATH 472 Financial Mathematics

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2018
Introduction

Definition
Hedging is the practice of making a portfolio of investments less sensitive to changes in market variables.

There are various hedging strategies.

During this discussion we will explore, Delta hedging which attempts to keep the $\Delta$ of a portfolio nearly 0, so that the value of the portfolio is insensitive to changes in the price of a security.
Responsibilities of the Seller of an Option

- A financial institution sells a call option on a security to an investor.

- If, at expiry, the market price of the security is below the strike price, the call option will not be exercised and the financial institution keeps the premium paid on the call by the investor.

- If, at expiry, the market price of the security exceeds the strike price, the financial institution must ensure the investor can purchase the security at the strike price. How?
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- If, at expiry, the market price of the security exceeds the strike price, the financial institution must ensure the investor can purchase the security at the strike price.
  How?
- The financial institution must sell the security for the strike price to the investor.
A bank sells 100 European calls on a security where $S_0 = $50, $K = $52, $r = 2.5\%$, $T = 4/12$, and $\sigma = 22.5\%$. According to the Black-Scholes option pricing formula, $C^e = $1.91965.

The financial institution may borrow funds to purchase 100 shares of the security. This is called a covered position.

At expiry the profit is $100 \left( S_T - \left( S_T - 52 \right) - 1.91965 \right) e^{-0.025 \left( 4/12 \right)}$.

Explain the meanings of the terms in the expression above.
A bank sells 100 European calls on a security where $S_0 = $50, $K = $52, $r = 2.5\%$, $T = 4/12$, and $\sigma = 22.5\%$. According to the Black-Scholes option pricing formula, $C^e = $1.91965.

- The financial institution may borrow funds to purchase 100 shares of the security. This is called a covered position.
- At expiry the profit is

$$100(S_T - (S_T - 52)^+ - (50 - 1.91965)e^{0.025(4/12)}).$$

- Explain the meanings of the terms in the expression above.
If $S_T \geq 52$ the cashflow is $351.73$.

If $S_T \approx 48.4827$ the cashflow is zero.

If $S_T = 46$ the cashflow is $-248.27$.

In the worst case of $S_T = 0$ the cashflow is $-4848.27$. 

Covered Strategy (2 of 2)
As an alternative to the covered strategy, the financial institution may wait until expiry to purchase the 100 shares of the security. It would then immediately sell the shares to the investor. This is called a **naked position**.

At expiry the profit to the financial institution is

$$100(1.91965e^{0.025(4/12)} - (S_T - 52)^+).$$
If $S_T \leq 52$ the profit is $193.71$.

Profit is zero when $S_T \approx 53.9357$.

If $S_T = 56$ the profit is $-206.43$.

The losses to the financial institution are unbounded as $S_T \to \infty$. 
Stop-Loss Strategy

- Suppose the financial institution sells a $K$-strike call option on a security.
- The financial institution will buy the security whenever $S_t \geq K$ and will sell it when $S_t < K$.
- The financial institution wants a covered position whenever the call option may be exercised and a naked position whenever it will not be exercised.
Illustration

The diagram shows a stochastic process $S_t$ over time $t$, with a threshold $K$.
Drawbacks of Stop-Loss Strategy

**Question:** why is the stop-loss strategy ineffective in practice?

- **Cost of setting up this strategy** $(S_0 - K) +$.
- Purchases and sales of security for $t > 0$ must be present valued.
- Purchases and sales cannot be made exactly at price $K$. Purchases will be made at price $K + \delta$ and sales as $K - \delta$ for some $\delta > 0$.
- As $\delta \to 0^+$ the number of purchases and sales will grow unbounded.
- Strategy ignores transaction costs.
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- As \( \delta \to 0^+ \) the number of purchases and sales will grow unbounded.
- Strategy ignores transaction costs.
Delta Hedging

**Recall:** If the value of a solution to the Black-Scholes PDE is $F$ then $\Delta = \frac{\partial F}{\partial S}$ where $S$ is the value of some security underlying $F$.

- If $F$ is an option then for every unit change in the value of the underlying security, the value of the option changes by approximately $\Delta$.
- A portfolio consisting of securities and options is called **Delta-neutral** if for every call option sold, $\Delta$ units of the security are bought.
Example of Delta Hedging

- Suppose $S_0 = $90, $r = 10\%$, $\sigma = 50\%$, $K = $95, and $T = 1$.

- Under these conditions $w = 0.341866$, the value of a European call option is $C^e = 19.4603$ and Delta for the option is 

$$\Delta = \frac{\partial C^e}{\partial S} = \Phi (w) = 0.633774.$$ 

- If a financial institution sold 100 call options, the firm would receive $1946.03$ and would purchase 

$$(0.633774 \cdot 100) \cdot 90 = $5703.97$$

worth of the security.

- The financial institution will finance the security purchase by borrowing 

$$5703.97 - 1946.03 = $3757.94.$$
The value of the portfolio consisting of a short position in 100 European call options and a long position in 100 Δ shares of the underlying security is $3757.94.

Delta hedging this portfolio and periodically rebalancing it, should preserve its value.
Rebalancing a Portfolio

- If, after setting up the hedge, a financial institution does nothing else until expiry, this is called **static hedge** or a “hedge and forget” strategy.
- The value of the call option will decay as a function of time at the instantaneous rate $\Theta$.
- The price of the security will (probably) change during the life of the option, so the firm may choose to make periodic adjustments to the number shares of the security it holds. This is called a **dynamic hedge**.

This activity is known as **rebalancing** the portfolio.
Static Hedge

\[ \text{profit} = ((\Delta)S_0 - C^e)e^{rT} + (\Delta)S_T - (S_T - K)^+ \]
Suppose $S_0 = $90, $r = 10\%$, $\sigma = 50\%$, $K = $95, and $T = 1$. 

**Delta of Call**

<table>
<thead>
<tr>
<th>$S_T$</th>
<th>Δ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.4</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8</td>
</tr>
<tr>
<td>0.6</td>
<td>1.0</td>
</tr>
</tbody>
</table>

**Delta of Put**

<table>
<thead>
<tr>
<th>$S_T$</th>
<th>Δ</th>
</tr>
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<tbody>
<tr>
<td>-1.0</td>
<td>-0.8</td>
</tr>
<tr>
<td>-0.8</td>
<td>-0.6</td>
</tr>
<tr>
<td>-0.6</td>
<td>-0.4</td>
</tr>
<tr>
<td>-0.4</td>
<td>-0.2</td>
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<tr>
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</table>
Delta and “Money-ness”

\[ \Delta \]

\[ S_t < K \]
\[ S_t = K \]
\[ S_t > K \]
Extended Example

Assume the value of the security follows the random walk shown below.

The European call option will be exercised since $S_T > 95$. 
End of First Month

- Suppose that $S_{1/12} = 90.56$.
- The number of options sold remains constant ($n = 100$), but the value of the options has changed.

\[ C^e(S_{1/12}, 1/12) = 18.7736 \]
End of First Month

- Suppose that $S_{1/12} = 90.56$.
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$$C^e(S_{1/12}, 1/12) = 18.7736$$

- **Question**: if the financial institution liquidated their position by selling their stock and re-purchasing the options, would the financial institution make or lose money during the first month?

Gain/Loss on Security

$$100 \cdot 0.63374 \times (90.56 - 90) = 35.4913$$

Gain/Loss on Option

$$100 \cdot (19.4603 - 18.7736) = 68.6672$$

Interest

$$-3757.94 \times (e^{0.10/12} - 1) = -31.447$$

Profit

$$72.7115$$
End of First Month

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<tbody>
<tr>
<td>Gain/Loss on Security</td>
<td>$100 \cdot 0.633774(90.56 - 90) = 35.4913$</td>
<td>$\text{USD}$</td>
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<td>Gain/Loss on Option</td>
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Rebalancing at End of First Month

- Re-compute $\Delta$ using $S_{1/12}$ and $t = 1/12$.

\[
\Delta = \Phi (w) = 0.629624
\]

- The current value of $\Delta$ is smaller than the original value. The financial institution may sell $(0.633774 - 0.629624) \times 100 = 0.415$ shares of the security at the current price of $S_{1/12} = 90.56$.

- This generates a cashflow of $(0.415)(90.56) = 37.5824$. For the second month the financial institution owns $100 \times 0.629624 = 62.9624$ shares of the security.
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End of Second Month

- Suppose that $S_{2/12} = $91.25.
- The value of the options has changed.

$$C^e(S_{2/12}, 2/12) = 18.1189$$

- Outstanding balance on loan,

$$3757.94 + 31.447 − 37.5824 = $3751.80.$$
End of Second Month

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$$ 3757.94 + 31.447 - 37.5824 = $3751.80. $$

- **Question**: if the financial institution liquidated their position by selling their stock and re-purchasing the options, would the financial institution make or lose money during the second month?

$$ \text{Gain/Loss on Security} = 100 \cdot 0.629624 \left( 91.25 - 90.56 \right) = $43.4441 $$

$$ \text{Gain/Loss on Option} = 100 \cdot \left( 18.7736 - 18.1189 \right) = $65.47 $$

$$ \text{Interest} = -3751.80 \left( e^{0.10/12} - 1 \right) = -$31.3956 $$

$$ \text{Profit} = $77.5185 $$
End of Second Month

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</tr>
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</tr>
<tr>
<td>Profit</td>
<td>$77.5185$</td>
</tr>
</tbody>
</table>
Rebalancing at End of Second Month

- Re-compute $\Delta$ using $S_{2/12}$ and $t = 2/12$.

$$\Delta = \Phi (w) = 0.626484$$

- The current value of $\Delta$ is smaller than the previous value.
- The financial institution may sell $(0.629624 - 0.626484) \cdot 100 = 0.314$ shares of the security at the current price of $S_{2/12} = 91.25$.
- This generates $(0.314)(91.25) = 28.6525$ in cashflow which repays a portion of the loan.
- For the third month the financial institution owns $100 \cdot 0.626484 = 62.6484$ shares of the security.
Rebalancing at End of Second Month

▶ Re-compute $\Delta$ using $S_{2/12}$ and $t = 2/12$.

$$\Delta = \Phi (w) = 0.626484$$

▶ The current value of $\Delta$ is smaller than the previous value. The financial institution may sell

$$(0.629624 - 0.626484) \cdot 100 = 0.314$$

shares of the security at the current price of $S_{2/12} = $91.25.

▶ This generates $(0.314)(91.25) = $28.6525 in cashflow which repays a portion of the loan.
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- For the third month the financial institution owns
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  100 \cdot 0.626484 = 62.6484
  \]
  shares of the security.
The end-of-the-month profit (or loss) shows the amount of money the financial institution can take from (or must put in to) the portfolio.

We will assume the financial institution can always borrow up to the value of the shares of the security in the portfolio.

There are three cashflow streams in to/out of the portfolio:
- Borrowing/Repaying the loan,
- Purchasing/Selling the security,
- Interest charges on outstanding balance of the loan.
For each month \( i = 0, 1, \cdots, 12 \) define the following quantities:

- \( S_i \), market price of the security.
- \( \Delta_i \), Delta of call option.
- \( N_i \), the number of shares of security purchased at beginning of the month.
- \( \text{Cost}_i \), cost of securities purchased at the beginning of the month.
- \( \text{CC}_i \), cumulative cost of securities purchased including interest.
If $n$ options are sold then

\[ \begin{align*}
N_0 &= n \Delta_0 \\
N_i &= n(\Delta_i - \Delta_{i-1}) \text{ for } i \geq 1 \\
\text{Cost}_i &= N_i S_i = n(\Delta_i - \Delta_{i-1}) S_i \\
\text{CC}_i &= \sum_{k=0}^{i} (\text{Cost}_i) e^{r(i-k)/12}
\end{align*} \]
<table>
<thead>
<tr>
<th>Month</th>
<th>Stock Price</th>
<th>$\Delta_i$</th>
<th>Shares Purchased</th>
<th>Cost of Shares</th>
<th>Cumulative Cost</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>90.00</td>
<td>0.633774</td>
<td>63.3774</td>
<td>5703.97</td>
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</tr>
<tr>
<td>1</td>
<td>90.56</td>
<td>0.629624</td>
<td>−0.415022</td>
<td>−37.5844</td>
<td>5714.11</td>
</tr>
<tr>
<td>2</td>
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<td>−0.313942</td>
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<tr>
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<td>9</td>
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<td>5768.92</td>
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<td>96.21</td>
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<td>12</td>
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<td>1.000000</td>
<td>41.387400</td>
<td>3986.44</td>
<td>9822.64</td>
</tr>
</tbody>
</table>
Unwinding the Firm’s Position

- At expiry the financial institution sells the $n = 100$ shares of the security for $K = $95/share.
- The firm’s profit is the difference between the market value of the securities held, the outstanding balance on the loan, and the accumulated value of the value of the options sold.

$$100(95) - 9822.64 + 100(19.4603)e^{0.10} = $1828.06$$
Second Realization

Suppose the price of the security followed the path shown below.
<table>
<thead>
<tr>
<th>Month</th>
<th>Stock Price</th>
<th>$\Delta_i$</th>
<th>Shares Purchased</th>
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<td>63.3774</td>
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<tr>
<td>1</td>
<td>90.29</td>
<td>0.627266</td>
<td>-0.650845</td>
<td>-58.7648</td>
<td>5692.93</td>
</tr>
<tr>
<td>2</td>
<td>90.67</td>
<td>0.621182</td>
<td>-0.608371</td>
<td>-55.161</td>
<td>5685.41</td>
</tr>
<tr>
<td>3</td>
<td>91.04</td>
<td>0.61462</td>
<td>-0.656143</td>
<td>-59.7352</td>
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<tr>
<td>4</td>
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<td>5668.69</td>
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<tr>
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<tr>
<td>7</td>
<td>93.00</td>
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<tr>
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<td>0.587286</td>
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<td>95.26</td>
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<td>-37.8874</td>
<td>5665.11</td>
</tr>
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<td>12</td>
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</table>

Complete the rebalancing table.
End of the 12th Month (Expiry)

<table>
<thead>
<tr>
<th>Month</th>
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<th>( \Delta_i )</th>
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<td>-59.7352</td>
<td>5673.25</td>
</tr>
<tr>
<td>4</td>
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<td>-0.568502</td>
<td>-52.0407</td>
<td>5668.69</td>
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<td>-0.529408</td>
<td>-49.2349</td>
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<td>0.587286</td>
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<td>-14.5443</td>
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<tr>
<td>9</td>
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<td>0.582103</td>
<td>-0.518311</td>
<td>-49.0218</td>
<td>5655.67</td>
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<tr>
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<td>0.578126</td>
<td>-0.397726</td>
<td>-37.8874</td>
<td>5665.11</td>
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<tr>
<td>11</td>
<td>94.30</td>
<td>0.531350</td>
<td>-4.67754</td>
<td>-441.092</td>
<td>5271.43</td>
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<tr>
<td>12</td>
<td>93.20</td>
<td>0.000000</td>
<td>-53.13500</td>
<td>-4952.19</td>
<td>363.353</td>
</tr>
</tbody>
</table>
At expiry the financial institution has hold no shares of the security.

Note that the firm’s profit is the difference between the market value of the securities held, the cumulative cost of the hedge, and the accumulated value of the options sold.

\[
0000.00 - 363.353 + 100(19.4603)e^{0.10} = \$2514.05
\]
Self-Financing Portfolios

Definition
A portfolio consisting of a sold call $C(S, t)$ and a long position in $\Delta$ shares of the underlying security is said to be self-financing if the profit/loss from a movement in stock price is zero.
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Suppose the risk-free interest rate is $r$ and the volatility of the security is $\sigma$. 
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- Consider a sold $K$-strike European call on a security whose current value is $S$ and purchased $\Delta$ shares of the security.
- Suppose the risk-free interest rate is $r$ and the volatility of the security is $\sigma$.
- **Question**: what moves in security price result in a self-financing portfolio?
Let $K = 100$, $S = 100$, $r = 0.10$, $\sigma = 0.50$, and $T = 1$. The one-day profit curve resembles that shown below.
The self-financing one-day movements in the price of the security are the solutions to the equation:

\[ V_1 - V_0 - S_1(\Delta_1 - \Delta_0) - V_0(e^{r/365} - 1) = 0. \]
Numerical Example (2 of 2)

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\[ V_1 - V_0 - S_1(\Delta_1 - \Delta_0) - V_0(e^{r/365} - 1) = 0. \]

Numerically these roots are estimated to be

\[ S_1 = 99.1748 \quad \text{and} \quad S_1 = 100.829. \]
Other Solutions to the Black-Scholes PDE

We have already seen that the values of European Call and Put options satisfy the Black-Scholes PDE.

\[ rF = F_t + \frac{1}{2} \sigma^2 S^2 F_{SS} + rSF_S \]

Other financial instruments solve the PDE as well (but satisfy different boundary and/or final conditions than the options).
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Show that the following are also solutions.

1. \( F(S, t) = S \)
2. \( F(S, t) = Ae^{rt} \)
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Show that the following are also solutions.

1. \( F(S, t) = S \)
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Hence, the security itself and cash are both solutions to the Black-Scholes PDE.
Delta Neutral Portfolios

A portfolio consists of a short position in a European call option and a long position in the security (Delta hedged). Thus the net value $P$ of the portfolio is

$$P = C - (\Delta)S = C - \frac{\partial C}{\partial S}\bigg|_{S_0} S.$$

$P$ satisfies the Black-Scholes equation since $C$ and $S$ separately solve it. Thus Delta for the portfolio is

$$\frac{\partial P}{\partial S} = \frac{\partial C}{\partial S} - \frac{\partial C}{\partial S}\bigg|_{S_0}.$$

$$\frac{\partial P}{\partial S} \approx 0 \text{ when } S \approx S_0.$$
Taylor Series for $\mathcal{P}$

\[
\mathcal{P} = \mathcal{P}_0 + \frac{\partial \mathcal{P}}{\partial t}(t - t_0) + \frac{\partial \mathcal{P}}{\partial S}(S - S_0) + \frac{\partial^2 \mathcal{P}}{\partial S^2} \frac{(S - S_0)^2}{2} + \cdots
\]

\[
\delta \mathcal{P} = \Theta \delta t + \Delta \delta S + \frac{1}{2} \Gamma (\delta S)^2 + \cdots
\]

\[
\delta \mathcal{P} \approx \Theta \delta t + \frac{1}{2} \Gamma (\delta S)^2
\]

- $\Theta$ is not stochastic and thus must be retained.
- What about $\Gamma$?
Recall: \( \Gamma = \frac{\partial^2 F}{\partial S^2} \)

- Since \( \frac{\partial^2}{\partial S^2} (S) = 0 \) a portfolio cannot be made gamma neutral if it contains only an option and its underlying security.
- Portfolio must include an additional component which depends non-linearly on \( S \).
- Portfolio can include two (or more) different types of option dependent on the same security.
Example (1 of 5)

- Suppose a portfolio contains options with two different strike times written on the same security.
- A firm may sell European call options with a strike time three months and buy European call options on the same security with a strike time of six months.
- Let the number of the early options sold be $n_e$ and the number of the later options purchased be $n_l$. 

\[
\Gamma_p = n_e \Gamma_e - n_l \Gamma_l
\]
Example (1 of 5)

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- Let the number of the early options sold be $n_e$ and the number of the later options purchased be $n_l$.

The Gamma of the portfolio would be

$$\Gamma_P = n_e \Gamma_e - n_l \Gamma_l,$$

where $\Gamma_e$ and $\Gamma_l$ denote the Gammas of the earlier and later options respectively.
Example (2 of 5)

- Choose $n_e$ and $n_l$ so that $\Gamma_P = 0$.
- Introduce the security so as to make the portfolio Delta neutral.
- **Question:** Why does changing the number of shares of the security in the portfolio affect $\Delta$ but not $\Gamma$?
Example (2 of 5)

- Choose \( n_e \) and \( n_l \) so that \( \Gamma_P = 0 \).
- Introduce the security so as to make the portfolio Delta neutral.
- **Question:** Why does changing the number of shares of the security in the portfolio affect \( \Delta \) but not \( \Gamma \) ?

With the proper values of \( n_e \) and \( n_l \) then

\[
\delta P \approx \Theta \delta t.
\]
Suppose $S = $100, $\sigma = 0.22$, and $r = 2.5\%$.

An investment firm sells a European call option on this security with $T_e = 1/4$ and $K = $102.

The firm buys European call options on the same security with the same strike price but with $T_l = 1/2$.

Gamma of the 3-month option is $\Gamma_e = 0.03618$ and Gamma of the 6-month option is $\Gamma_l = 0.02563$.

The portfolio is Gamma neutral in the first quadrant of $n_e n_l$-space where the equation

$$0.03618 n_e - 0.02563 n_l = 0$$

is satisfied.
Suppose \( n_e = 100000 \) of the three-month option were sold.

Portfolio is Gamma neutral if \( n_l = 141163 \) six-month options are purchased.

Before including the underlying security in the portfolio, the Delta of the portfolio is

\[
ne \Delta_e - nl \Delta_l = (100000)(0.4728) - (141163)(0.5123) = -25038.
\]

Portfolio can be made Delta neutral if 25,038 shares of the underlying security are sold short.
Over a wide range of values for the underlying security, the value of the portfolio remains nearly constant.
Conclusion

- Rho and Vega can be used to hedge portfolios against changes in the interest rate and volatility respectively.
- We have assumed that the necessary options and securities could be bought or sold so as to form the desired hedge.
- If this is not true then a firm or investor may have to substitute a different, but related security or other financial instrument in order to set up the hedge.
Homework

- Read Sections x.y
- Exercises:
These slides are adapted from the textbook,

_An Undergraduate Introduction to Financial Mathematics_,

author: J. Robert Buchanan


address: 27 Warren St., Suite 401–402, Hackensack, NJ 07601

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