Please solve the following problems dealing with topics in elementary probability. Show all work and answers on your own paper. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Your results and supporting work are due at class time on Tuesday, August 2, 2005.

1. Suppose that $A$ is an event such that $P(A) = 0$ and that $B$ is any other event. Prove that $A$ and $B$ are independent events.

   \[ P(A \cap B) = P(B)P(A|B) = P(B) \cdot 0 \]

   We can also see that $P(A)P(B) = 0$, thus together

   \[ P(A \cap B) = 0 = P(A)P(B) \]

   which means events $A$ and $B$ are independent.

2. Suppose that 10000 tickets are sold in one lottery and that 5000 tickets are sold in another lottery. If a man owns 100 tickets in each lottery, what is the probability that he will win at least one first prize?

   Let $A$ be the outcome that the man wins the first lottery and let $B$ be the outcome that the man wins the second lottery.

   \[ P(A) = \frac{100}{10000} = \frac{1}{100} \quad \text{and} \quad P(B) = \frac{100}{5000} = \frac{1}{50} \]

   Winning both lotteries is not mutually exclusive, but we will assume the lotteries are independent, thus by the addition rule for probabilities

   \[ P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) = \frac{1}{100} + \frac{1}{50} - \frac{1}{5000} = \frac{149}{5000} \approx 0.0298. \]

3. In the World Series of baseball two teams $A$ and $B$ play a sequence of games against each other, and the first team that wins a total of four games becomes the winner of the World Series. If the probability that team $A$ will win any particular game against team $B$ is $49/100$, what is the probability that team $A$ will win the World Series?

   Team $A$ may win the series in either 4 games, 5 games, 6 games, or 7 games. These are mutually exclusive alternatives. If team $A$ wins the World Series then it must win the last game played. Using the formula for binomial random variables then the probability of team $A$ winning 4 of $k$ games is

   \[ \frac{(k-1)!}{3!(k-4)!} \left( \frac{49}{100} \right)^4 \left( 1 - \frac{49}{100} \right)^{k-4}. \]

   Thus the probability that team $A$ wins the World Series is

   \[ P(A \text{ wins}) = (0.49)^4 + 4(0.49)^4(0.51) + 10(0.49)^4(0.51)^2 + 20(0.49)^4(0.51)^3 \approx 0.478134 \]
4. If \( A \) and \( B \) are independent events and \( 0 < P(B) < 1 \), what is the value of \( P(A^c | B^c) \)?

\[
P(A^c | B^c) = \frac{P(A^c \cap B^c)}{P(B^c)} \]
\[
= \frac{\frac{P((A \cup B)^c)}{P(B)}}{1 - P(B)}
\]
\[
= \frac{1 - (P(A) + P(B) - P(A)P(B))}{1 - P(B)}
\]
\[
= \frac{1 - P(A) - P(B)(1 - P(A))}{1 - P(B)}
\]
\[
= \frac{(1 - P(A))(1 - P(B))}{1 - P(B)}
\]
\[
= 1 - P(A)
\]
\[
P(A^c)
\]

5. A box contains three cards. One card is red on both sides, one card is green on both sides, and one card is red on one side and green on the other. One card is selected at random from a box, and the color on one side is observed. If this side is green, what is the probability that the other side is also green?

The probability of interest is the probability that the selected card is the card which is green on both sides. We have observed that the card is green on one side. We can express the answer as a conditional probability.

\[
P(\text{green} | \text{green}) = \frac{P(\text{green} \cap \text{green})}{P(\text{green})} \]
\[
= \frac{1/3}{1/2} \]
\[
= \frac{2}{3}
\]

6. Suppose that the probability density function of a random variable \( X \) is given by

\[
f(x) = \begin{cases} 
  0 & \text{for } x < 0 \\
  e^{-x} & \text{for } x \geq 0 
\end{cases}
\]

Find the \( P(X \geq 2) \).

\[
P(X \geq 2) = 1 - P(X < 2) = 1 - \int_{-\infty}^{4} e^{-x} \, dx
\]
\[
= 1 - \left[ -e^{-x} \right]_{-\infty}^{4}
\]
\[
= 1 + e^{-4}
\]
\[
1 + e^{-2} - 1 = e^{-2} \approx 0.135335
\]

7. If an integer between 1 and 100 inclusive is to be chosen at random, what is the expected value of the chosen integer?

We observe that if the random variable \(X\) represents the integer chosen then \(P(X = x) = 1/100\) for \(x = 1, 2, \ldots, 100\). Thus

\[
E[X] = \sum_{x=1}^{100} x P(X = x) = \frac{1}{100} \sum_{x=1}^{100} x = \frac{1}{100} \cdot \frac{100(101)}{2} = \frac{5050}{100} = 50.5
\]

8. Suppose that the random variable \(X\) has a uniform distribution on the interval \((0, 1)\) and that \(Y\) is a random variable with a uniform distribution on the interval \((5, 9)\) and the \(X\) and \(Y\) are independent. Suppose also that a rectangle is to be constructed for which the lengths of the two adjacent sides are \(X\) and \(Y\). What is the expected value of the area of the rectangle?

The probability distribution function for \(X\) is the function

\[
f(x) = \begin{cases} 
1 & \text{if } 0 \leq x \leq 1, \\
0 & \text{otherwise}.
\end{cases}
\]

The probability distribution function for \(Y\) is the function

\[
g(y) = \begin{cases} 
\frac{1}{4} & \text{if } 5 \leq y \leq 9, \\
0 & \text{otherwise}.
\end{cases}
\]

Since the random variables are independent, joint probability distribution of the ordered pair of random variables \((X, Y)\) is \(f(x)g(y)\). The expected value of the area of the rectangle is then

\[
E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x)g(y) \, dx \, dy
= \left( \int_{0}^{1} x \, dx \right) \left( \int_{5}^{9} \frac{y}{4} \, dy \right)
= \left( \frac{1}{2} \right) \left( \frac{y^2}{8} \right)_{5}^{9}
= \frac{y^2}{16} \left| _{5}^{9} \right.
= \frac{81 - 25}{16}
= 3.5
\]