Linearity vs. Nonlinearity

Millersville University Scholarship Social

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March 25, 2005

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Notions of Linearity

Spaces, operators, and functions are at times given the designation of “linear” by mathematicians.

A linear space consists of a set of vectors \( V \), a field of scalars \( F \), a rule for addition of vectors, and a rule for multiplication of scalars and vectors, satisfying a set of axioms.

An operator \( L \), is linear if for all vectors \( x \) and \( y \) and scalars \( c \) the following relationship is true.

\[
L(cx + y) = cL(x) + L(y)
\]
Example of Linearity

- Field: rational numbers \( \{p/q : p, q \in \mathbb{Z}, \gcd(p, q) = 1\} \)
- Vectors: \( \{1, x, x^2, x^3\} \)
- Vector addition: conventional +
- Scalar multiplication: conventional ·

Linear space: space of all polynomials of degree 3 or less with rational coefficients.

Linear operator: differentiation,

\[
D(1) = 0, \quad D(x) = 1, \quad D(x^2) = 2x, \quad D(x^3) = 3x^2.
\]
Example (cont.)

\[
D \left( \frac{3}{2}x^2 + 5x + 11 \right) = D \left( \frac{3}{2}x^2 \right) + D(5x) + D(11)
\]

\[
= \frac{3}{2}D(x^2) + 5D(x) + 11D(1)
\]

\[
= \frac{3}{2}(2x) + 5(1) + 11(0)
\]

\[
= 3x + 5
\]
Utility of Linearity

Axiomatically:

- Linear spaces are closed under vector addition and scalar multiplication.
- Linear operators “commute” with vector addition and scalar multiplication.
Heat equation

\[ u_t = ku_{xx} \quad \text{for} \ 0 < x < L, \ t > 0, \]
\[ u_x(0, t) = u(L, t) = 0 \quad \text{for} \ t > 0 \]

If \( n \in \mathbb{N} \) then

\[ u_n(x, t) = e^{-\frac{(4n^2 - 4n + 1)\pi^2 t}{4L^2}} \cos \left( \frac{(2n - 1)\pi x}{2L} \right) \]

solves this boundary value problem. \( \{u_n(x, t)\}_{n=1}^{\infty} \) forms a linear space and thus sums and scalar products also solve the boundary value problem.

Basis of Fourier Series.
Linear Functions

Objective: keep mathematical prerequisites minimal.
High school algebra: $y = mx + b$

- slope of the line $m$
- $y$-intercept of the line $b$
Linear Maps

**Map:** rule of correspondence between an input value and a resulting output value. AKA, a function.

**Initial condition:** starting point for the iteration.

**Orbit:** list of output values of the iterated map.

**Prototype:** \( x \rightarrow \alpha x \) with \( x_0 = z \) where \( \alpha \) is any scalar.

**Example:** \( x \rightarrow \frac{1}{2} x \) with \( x_0 = 1 \)
Example: $x \rightarrow x/2$ with $x_0 = 1$ produces orbit:

$$\left\{ 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots \right\}.$$
Qualitative Dynamics

Concepts of interest:

**Equilibria:** agreement between input and output values of the map.
AKA, **fixed points**.

**Periodic orbits:** repeated cycling of output values of the map.

**Stability:** does the orbit change fundamentally if the initial condition is changed by a small amount?
Linear Dynamics

Complex number $\alpha = x + iy = re^{i\theta}$ where

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan \frac{y}{x}.$$ 

Linear map: $z \rightarrow \alpha z = re^{i\theta} z$

- If $r < 1$, all orbits approach 0.
- If $r > 1$, all orbits approach $\infty$.
- If $r = 1$ and $\theta/\pi \in \mathbb{Q}$, all orbits are periodic.
- Otherwise, all orbits are bounded, but not periodic.
Orbit: \( \{ z, r e^{i\theta} z, r^2 e^{i2\theta} z, r^3 e^{i3\theta} z, \ldots, r^n e^{in\theta} z, \ldots \} \)

\[ r < 1 \quad \text{and} \quad r > 1 \]

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Case: $r = 1$

Orbit: $\{ z, e^{i\theta}z, e^{i2\theta}z, e^{i3\theta}z, \ldots, e^{in\theta}z, \ldots \}$

$\theta = \pi/30$

$\theta = 1/30$
Nonlinearity

- “Tautology”: any mapping which is not linear is nonlinear.

- Logistic map: \( x \rightarrow \alpha x(1 - x) \)
  - The dynamics of the logistic map can be very complicated depending on the initial condition and the parameter \( \alpha \).
  - For \( 0 \leq \alpha \leq 4 \), the logistic map maps \([0, 1]\) into itself.
Trivial Fixed Point

Consider the map: $x \rightarrow 0.9x(1 - x)$. 
Consider the map: \( x \rightarrow 1.9x(1 - x) \).
Consider the map: $x \rightarrow 3.1x(1 - x)$.

**Equilibria:**

$$x = 3.1x(1 - x) = 3.1x - 3.1x^2$$

$$0 = 2.1x - 3.1x^2 = x(2.1 - 3.1x)$$

$x = 0$ or $x = 2.1/3.1 = 21/31 \approx 0.677419$.

**Periodic orbits:** since

$$\frac{41 - \sqrt{41}}{62} \rightarrow \frac{41 + \sqrt{41}}{62} \quad \text{and} \quad \frac{41 + \sqrt{41}}{62} \rightarrow \frac{41 - \sqrt{41}}{62}$$

$\{(41 - \sqrt{41})/62, (41 + \sqrt{41})/62\}$ is a periodic orbit of length 2, sometimes called a 2-cycle.
2-Cycle

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Consider the map: 
\[ x \rightarrow 3.5x(1 - x). \]

**Equilibria:** \( x = 0 \) and \( x = 5/7 \) (both unstable)

**2-cycle:** \( 3/7 \rightarrow 6/7 \rightarrow 3/7 \) (unstable)

**4-cycle:**

\[
\begin{align*}
0.500884 & \rightarrow 0.874997 \\
& \rightarrow 0.38282 \\
& \rightarrow 0.826941 \\
& \rightarrow 0.500884
\end{align*}
\]
The logistic map undergoes a sequence of **period doubling** bifurcations as $\alpha$ increases.
A 3-cycle would be the solution to the following system of equations:

\[
\begin{align*}
  x_2 &= \alpha x_1 (1 - x_1) \\
  x_3 &= \alpha x_2 (1 - x_2) \\
  x_1 &= \alpha x_3 (1 - x_3) \\
  1 &= \alpha^3 (1 - 2x_1)(1 - 2x_2)(1 - 2x_3)
\end{align*}
\]

Solution:

\[
\begin{align*}
  \alpha &= 1 + 2\sqrt{2} \\
  x_1 &\approx 0.956318 \\
  x_2 &\approx 0.159929 \\
  x_3 &\approx 0.514355
\end{align*}
\]
Example of 3-cycle
Period 3 implies chaos.

Let $f(x)$ be a continuous, real-valued function mapping an interval into itself. If there exists a 3-cycle, then there exist cycles of all orders.

Period 3 implies period 5
Šarkovs’kii’s Theorem

$f(x)$ is a continuous, real-valued function on $\mathbb{R}$. Relation $\prec$ is defined for $\mathbb{N}$ as: $m \prec n$ if the existence of an $m$-cycle implies the existence of an $n$-cycle, but not conversely. $\mathbb{N}$ is ordered by $\prec$ and

\[3 \prec 5 \prec 7 \prec \cdots \prec 3 \cdot 2 \prec 5 \cdot 2 \prec 7 \cdot 2\]
\[\prec \cdots \prec 3 \cdot 2^2 \prec 5 \cdot 2^2 \prec 7 \cdot 2^2\]
\[\prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1.\]

Utility of Linearity

- Linear maps, operators, and functions are simple and have mostly predictable behavior.
- More complex operators can often be approximated by linear counterparts.
- The approximation is only locally valid.
Further Magnification
Root Finding

- Solving nonlinear equations is exceedingly difficult.
- Use the tangent line as a surrogate for the nonlinear part of the equation.

Given (nonlinear) \( f(x) = 0 \) and \( x_0 \) such that \( f(x_0) \approx 0 \) then near \( x_0 \) the linear function

\[
L(x) = f'(x_0)(x - x_0) + f(x_0) \approx f(x).
\]

\[
f'(x_0)(x - x_0) + f(x_0) = 0
\]
\[
f'(x_0)(x - x_0) = -f(x_0)
\]
\[
x - x_0 = -\frac{f(x_0)}{f'(x_0)}
\]

\[
x = x_0 - \frac{f(x_0)}{f'(x_0)}
\]
Newton’s Method

Iterative technique for approximating the solutions of nonlinear equations

Nonlinear mapping: \( x \rightarrow x - \frac{f(x)}{f'(x)} \)

Quadratically convergent

Example: \( \cos(\pi x^2) - \frac{x}{2} = 0. \)

\( x_0 = 0.5 \rightarrow 0.67 \rightarrow 0.6322 \rightarrow 0.63077 \rightarrow 0.630767 \rightarrow \cdots \)
Complex Dynamics

Consider the equation:
$$(z^2 - 1)(z^2 + a^2) = 0 \text{ with } a > 0.$$  

Newton mapping:
$$z \rightarrow \frac{3z^4 + (a^2 - 1)z^2 + z^2}{4z^3 + 2(a^2 - 1)z}$$